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Monotonic Behavior of Positive Solutions for Semi-Linear Parabolic Equations with Uniformly Elliptic Non-Local Operators in Half-Space

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Abstract. We address the problem given by the following partial differential equation: some semi-Linear parabolic equations with uniformly elliptic non-local operators in Half-Space. Initially, we establish a generalized weighted average inequality and a maximum principle in unbounded domains, which are crucial for the sliding method. Then, we employ sliding to demonstrate the monotonicity of bounded positive solutions. In this paper, we will remove the monotonicity assumption of the kernel function $a(x)$ by using the sliding method. The techniques employed in the process of this method have applications to other problems related to uniformly elliptic operators.

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1 Introduction and main results

In this paper, our focus lies on investigating the monotonicity of positive solutions to the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t)+(-\Delta)_a^s u(x,t)=f(t,u(x,t)), & (x,t)\in\mathbb{R}_+^n\times\mathbb{R}, \\ u(x,t)>0, & (x,t)\in\mathbb{R}_+^n\times\mathbb{R}, \\ u(x,t)=0, & (x,t)\in(\mathbb{R}^n\backslash\mathbb{R}_+^n)\times\mathbb{R}, \end{cases} \tag{1.1}$$

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where the weighted fractional Laplacian $(-\Delta)^s_a$ represents a uniformly elliptic non-local operator, defined as the following weighted operator

$$(-\Delta)^s_a u(x,t) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{a(x-y)(u(x)-u(y))}{|x-y|^{n+2s}} \mathrm{d}y, \tag{1.2}$$

with $0<s<1$ and $0<A_1\le a(x)\le A_2$. Here, P. V. denotes the principal value of the integral. In order to make the integral on the right side of (1.2) well defined, we suppose $u\in C^{1,1}_{loc}(\mathbb{R}^n)\cap\mathcal{L}_{2s}$ with

$$\mathcal{L}_{2s} = \left\{ u(x)\in L^1_{loc}(\mathbb{R}^n) \,|\, \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} \mathrm{d}x < +\infty \right\}.$$

The first equation in problem (1.1) is a variant of the equation:

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(u(x,t)), \tag{1.3}$$

where $u(x,t)$ represents the chemical concentration, $f(u)$ characterizes the kinetics, and $(-\Delta)^s$ denotes the diffusion coefficient defined as [1]:

$$(-\Delta)^s u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \mathrm{d}y. \tag{1.4}$$

Equation (1.3), known as the fractional reaction-diffusion equation, is widely used to study anomalous diffusion due to its ability to capture various phenomena [2]. Several results, including monotonicity and symmetry of solutions to the reaction-diffusion equation with $(-\Delta)^s$, have been established by scholars. Corresponding results for local or non-local elliptic equations are obtained in [3–12]. For the study of positive solutions of the fractional parabolic equations with respect to space variables, one can refers to [13–21].

In contrast to $(-\Delta)^s$, the operator $(-\Delta)^s_a$ emerges from jump L'evy processes [22] and finds applications in stochastic control problems [23–26]. When $a=1$, $(-\Delta)^s_a$ reduces to the well-known fractional Laplacian $(-\Delta)^s$. Additionally, as s approaches 1, the fractional Laplacian $(-\Delta)^s$ converges to the standard Laplacian $-\Delta$ [24]. Caffarelli and Silvestre [27] has deduced the $C^{1,2s}$ regularity result for the purely non-local Isaacs equations by the method of compactness and perturbation, where the Isaacs equations are in the form of

$$(-\Delta)^s_a u(x) = f(u),$$

with $a(x,y)$ satisfying $0<A_1\le a(x)\le A_2$, $\nabla_y a\le C|y|^{-1}$ and be continuous in x for a modulus of continuity independent of y.

Previous work by [15] established the monotonicity of positive solutions for (1.3) using the moving plane method. However, for problem (1.1), due to the monotonicity requirement on the operator kernel $a(x-y)/|x-y|^{n+2s}$, one needs to assume that