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Monotonic Behavior of Positive Solutions for Semi-Linear Parabolic Equations with Uniformly Elliptic Non-Local Operators in Half-Space

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Abstract. We address the problem given by the following partial differential equation: some semi-Linear parabolic equations with uniformly elliptic non-local operators in Half-Space. Initially, we establish a generalized weighted average inequality and a maximum principle in unbounded domains, which are crucial for the sliding method. Then, we employ sliding to demonstrate the monotonicity of bounded positive solutions. In this paper, we will remove the monotonicity assumption of the kernel function $a(x)$ by using the sliding method. The techniques employed in the process of this method have applications to other problems related to uniformly elliptic operators.

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1 Introduction and main results

In this paper, our focus lies on investigating the monotonicity of positive solutions to the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t)+(-\Delta)_a^s u(x,t)=f(t,u(x,t)), & (x,t)\in\mathbb{R}_+^n\times\mathbb{R}, \\ u(x,t)>0, & (x,t)\in\mathbb{R}_+^n\times\mathbb{R}, \\ u(x,t)=0, & (x,t)\in(\mathbb{R}^n\backslash\mathbb{R}_+^n)\times\mathbb{R}, \end{cases} \tag{1.1}$$

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where the weighted fractional Laplacian $(-\Delta)_a^s$ represents a uniformly elliptic non-local operator, defined as the following weighted operator

$$(-\Delta)_a^s u(x,t) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{a(x-y)(u(x)-u(y))}{|x-y|^{n+2s}} \mathrm{d}y, \tag{1.2}$$

with $0<s<1$ and $0<A_1\le a(x)\le A_2$. Here, P. V. denotes the principal value of the integral. In order to make the integral on the right side of (1.2) well defined, we suppose $u \in C_{loc}^{1,1}(\mathbb{R}^n) \cap \mathcal{L}_{2s}$ with

$$\mathcal{L}_{2s} = \left\{ u(x) \in L_{loc}^1(\mathbb{R}^n) \,|\, \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} \mathrm{d}x < +\infty \right\}.$$

The first equation in problem (1.1) is a variant of the equation:

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(u(x,t)), \tag{1.3}$$

where $u(x,t)$ represents the chemical concentration, $f(u)$ characterizes the kinetics, and $(-\Delta)^s$ denotes the diffusion coefficient defined as [1]:

$$(-\Delta)^s u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \mathrm{d}y. \tag{1.4}$$

Equation (1.3), known as the fractional reaction-diffusion equation, is widely used to study anomalous diffusion due to its ability to capture various phenomena [2]. Several results, including monotonicity and symmetry of solutions to the reaction-diffusion equation with $(-\Delta)^s$, have been established by scholars. Corresponding results for local or non-local elliptic equations are obtained in [3–12]. For the study of positive solutions of the fractional parabolic equations with respect to space variables, one can refers to [13–21].

In contrast to $(-\Delta)^s$, the operator $(-\Delta)_a^s$ emerges from jump L'evy processes [22] and finds applications in stochastic control problems [23–26]. When $a=1$, $(-\Delta)_a^s$ reduces to the well-known fractional Laplacian $(-\Delta)^s$. Additionally, as s approaches 1, the fractional Laplacian $(-\Delta)^s$ converges to the standard Laplacian $-\Delta$ [24]. Caffarelli and Silvestre [27] has deduced the $C^{1,2s}$ regularity result for the purely non-local Isaacs equations by the method of compactness and perturbation, where the Isaacs equations are in the form of

$$(-\Delta)_a^s u(x) = f(u),$$

with $a(x,y)$ satisfying $0<A_1\le a(x)\le A_2$, $\nabla_y a \le C|y|^{-1}$ and be continuous in x for a modulus of continuity independent of y.

Previous work by [15] established the monotonicity of positive solutions for (1.3) using the moving plane method. However, for problem (1.1), due to the monotonicity requirement on the operator kernel $a(x-y)/|x-y|^{n+2s}$, one needs to assume that

$a(x-y)$ is monotonically decreasing with respect to $|x-y|$ when applying the moving plane method [24]. In contrast, the sliding method can overcome the limitation imposed on $(-\Delta)_a^s$. By employing the sliding method, Meng et al. [28] obtained monotonicity and non-existence results for solutions to semi-linear elliptic equations involving the operator $(-\Delta)_a^s$. However, to the best of our knowledge, there have been no monotonicity results available for the fractional parabolic problem involving $\partial_t+(-\Delta)_a^s$, except for the one-dimensional symmetry and monotonicity results for the fractional semi-linear parabolic equation obtained through the sliding method [16].

In order to use the sliding method, we first obtain the following generalized average inequalities and the maximum principle in unbounded domains.

Theorem 1.1 (A generalized weighted average inequality). *Let $u(x,t)\in C^1\big(\mathbb{R};C_{loc}^{1,1}(\mathbb{R}^n)\cap \mathcal{L}_{2s}\big)$. For each fixed $t\in\mathbb{R}$, if $u(x,t)$ attains its maximum at a point $\overline{x}$ in $\mathbb{R}^n$, then for any $r>0$, we have*

$$\frac{C_0}{C_{n,s}A_1}r^{2s}(-\Delta)_a^s u(\overline{x},t)+\int_{B_r^c(\overline{x})}u(y,t)\mathrm{d}\mu(y)\geq u(\overline{x},t), \tag{1.5}$$

where $B_r^c(\overline{x})$ is the complement of $B_r(\overline{x})$ in $\mathbb{R}^n$,

$$C_0=\frac{1}{\displaystyle\int_{B_1^c(0)}\frac{1}{|y|^{n+2s}}\mathrm{d}y}$$

is a positive constant, and

$$\int_{B_r^c(\overline{x})}\mathrm{d}\mu(y)=1.$$

Now, we establish a maximum principle in unbounded domains.

Theorem 1.2 (A maximum principle in unbounded domains). *Let $\Omega\subset\mathbb{R}^n$ be an open set, possibly unbounded and disconnected, and for any $x^0\in\mathbb{R}^n$ satisfying*

$$\underline{\lim}_{R\to\infty}\frac{|B_R(x^0)\cap\Omega^c|}{|B_R(x^0)|}>0. \tag{1.6}$$

Assume that $u(x,t)\in C^1\Big(\mathbb{R};C_{loc}^{1,1}(\mathbb{R}^n)\cap\mathcal{L}_{2s}\Big)$ is bounded from above, and satisfies

$$\begin{cases}\dfrac{\partial u}{\partial t}(x,t)+(-\Delta)_a^s u(x,t)\leq 0, & \text{at the points in } \Omega\times\mathbb{R} \text{ where } u(x,t)>0,\\ u(x,t)\leq 0, & \text{in } \Omega^c\times\mathbb{R}.\end{cases} \tag{1.7}$$

Then

$$u(x,t)\leq 0 \ \text{ in } \Omega\times\mathbb{R}. \tag{1.8}$$

Remark 1.1. The maximum principle is a basic factor to derive the monotonicity and symmetry of solutions by virtue of the sliding method. In [16], Chen and Wu deduced the monotonicity of entire positive solutions of the fractional parabolic equation

$$\frac{\partial u}{\partial t}(x,t)+(-\Delta)^s u(x,t)=f(u(x,t)), \quad (x,t)\in\mathbb{R}^n\times\mathbb{R}$$

where as the key ingredient, they established a generalized weighted average inequality and proved a maximum principle of unbounded regions. Here in this present paper, we extended these two theorems to the fractional parabolic equation involving a more general non-local operator $(-\Delta)_a^s$ in parallel.

Next, we consider the following problem

$$\begin{cases}\dfrac{\partial u}{\partial t}(x,t)+(-\Delta)_a^s u(x,t)=f(t,u(x,t)), & (x,t)\in\mathbb{R}_+^n\times\mathbb{R},\\ u(x,t)>0, & (x,t)\in\mathbb{R}_+^n\times\mathbb{R}.\end{cases} \tag{1.9}$$

To derive the monotonicity along x_n-direction, the following result is needed.

Theorem 1.3. *Let $u(x,t)\in C^1\left(\mathbb{R};C_{loc}^{1,1}(\mathbb{R}^n)\cap\mathcal{L}_{2s}\right)$ be a bounded solution of* (1.9)*. Assume $f(t,r)$ is continuous and satisfies **(a)** There exists $\mu>0$ such that for any $t\in\mathbb{R}$, $f(t,r)>0$ when $r\in(0,\mu)$, while $f(t,r)\leq 0$ when $r\geq\mu$.*
Suppose that $u(x,t)\leq\mu$ for $(x,t)\in(\mathbb{R}^n\backslash\mathbb{R}_+^n)\times\mathbb{R}$, then $u(x,t)\leq\mu$ for $(x,t)\in\mathbb{R}_+^n\times\mathbb{R}$.

Remark 1.2. Note that in [16], Chen assumed $u(x,t)<\mu$ directly. Moreover, since they were working on the whole space $\mathbb{R}^N$, it is not necessary to handle the regularity near the boundary, which is imperative in our problem on the half space.

Combining the generalized weighted average inequality with maximum principle in unbounded domains, we will apply the sliding method to derive the following monotonicity of the solutions for problem (1.5) based on the above theorem.

Theorem 1.4. *Let $u(x,t)\in C^1\left(\mathbb{R};C_{loc}^{1,1}(\mathbb{R}^n)\cap\mathcal{L}_{2s}\right)$ be a bounded solution of* (1.1) *satisfying*

$$\inf_{t\in\mathbb{R}} u(x,t)>0, \quad \forall x\in\mathbb{R}_+^n. \tag{1.10}$$

*$u(x',x_n,t)\xrightarrow{x_n\to+\infty}\mu$ uniformly in $t\in\mathbb{R}$. Assume that $\frac{\partial u}{\partial t}(x,t)$ is bounded, $f(t,r)$ is continuous and satisfies **(a)** and the following condition:*
***(b)** For any fixed $t\in\mathbb{R}$, $f(t,u)$ is non-increasing for $u\in[\mu-\delta,\mu]$ with some $\delta>0$.*
Then $u(x,t)$ is strictly monotone increasing in x_n.

Remark 1.3. The condition that $\frac{\partial u}{\partial t}(x,t)$ is bounded is only to ensure that u is uniformly continuous with respect to the time variable t (see the proof for more details). Therefore, this condition can actually be relaxed directly to the condition that u is uniformly continuous with respect to t, or can be replaced by any condition that can ensure such uniform continuity, such as the existence of the limit of solutions at infinity in t.

Remark 1.4. Various interesting solutions generated by the nonlinear term of the equation, i.e., pattern generation, have attracted a lot of attention. Classical problems such as $f(u)=u-u^3$ yields the Ginzburg-Landau equation [29], while $f(u)=u(1-u)(u-\alpha)$ with $0<\alpha<1$ yields the Zeldovich equation [30]. One can refer to [31] for the special case of $f(u)=u^2-u^3$. Note that the functions $f(u)=u-u^3$ and $f(u)=u^2-u^3$ are two typical examples of satisfying conditions **(a)** and **(b)**. If the function $f(t,u)$ at the right hand side of the equation monotonically decreases with respect to u, the situation becomes simple (one can refer to [16]). In addition, Chen etc. in [17] derived the monotonicity and non-existence of the solutions for the problem

$$\frac{\partial u}{\partial t}(x,t)+(-\Delta)^s u(x,t)=x_1 u^p(x,t), \quad (x,t)\in\mathbb{R}^n\times\mathbb{R},$$

where $0<s<1$ and $1<p<\infty$ by using the modified method of moving planes.

The rest of the paper is organized as follows. In section 2, we give the proof of the generalized weighted average inequality (Theorem 1.1) and the maximum principle in unbounded domains (Theorem 1.2). In addition, a comparison principle is also given in order to facilitate the later proofs. In section 3, we prove Theorems 1.3 and 1.4, showing the monotonicity by sliding.

2 The generalized average inequality, maximum principle in unbounded domains and comparison principle

2.1 The proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1: For a given t, suppose $u(x,t)$ gets the maximum at $\overline{x}$. Applying the definition of the fractional uniform elliptic operator, we get

$$\begin{aligned}(-\Delta)_a^s u(\overline{x},t) &= C_{n,s} P.V.\int_{\mathbb{R}^n} \frac{a(x-y)(u(\overline{x},t)-u(y,t))}{|\overline{x}-y|^{n+2s}}\mathrm{d}y \\ &\geq C_{n,s}\int_{B_r^c(\overline{x})} \frac{A_1(u(\overline{x},t)-u(y,t))}{|\overline{x}-y|^{n+2s}}\mathrm{d}y \\ &= C_{n,s}A_1 u(\overline{x},t)\int_{B_r^c(\overline{x})}\frac{1}{|\overline{x}-y|^{n+2s}}\mathrm{d}y - C_{n,s}A_1\int_{B_r^c(\overline{x})}\frac{u(y,t)}{|\overline{x}-y|^{n+2s}}\mathrm{d}y \\ &= \frac{C_{n,s}A_1}{C_0}\frac{u(\overline{x},t)}{r^{2s}} - C_{n,s}A_1\int_{B_r^c(\overline{x})}\frac{u(y,t)}{|\overline{x}-y|^{n+2s}}\mathrm{d}y.\end{aligned}$$

This shows that formula (1.5) is true and theorem 1.1 is proved. □

Proof of Theorem 1.2: Assume that (1.8) does not hold. Since $u(x,t)$ is bounded from above in $\Omega\times\mathbb{R}$, we can find a positive constant M such that

$$\sup_{(x,t)\in\Omega\times\mathbb{R}} u(x,t):=M>0.$$

The value of $u(x,t)$ in $\Omega\times\mathbb{R}$ is less than or equal to M, while the value of $u(x,t)$ outside the region $\Omega\times\mathbb{R}$ is less than 0.

The supremum of $u(x,t)$ may not be attained since the region $\Omega\times\mathbb{R}$ is unbounded, so it can be divided into two cases. In a relatively simple case, the value of $u(x,t)$ reaches M at a certain point. Set this point as $(\overline{x},\overline{t})$, so that $\frac{\partial u}{\partial t}(\overline{x},\overline{t})=0$ and hence

$$(-\Delta)_a^s u(x,t)\le 0.$$

From the condition on Ω (1.6), we get that

$$\int_{B_r^c(\overline{x})} u(y,t)\mathrm{d}\mu(y)$$

must be strictly less than M, which, by (1.5), means $u(\overline{x},\overline{t})$ is strictly less than M, a contradiction.

In the other case, the supremum M cannot be attained, and a sequence $(x^j,t_j)\subset\Omega\times\mathbb{R}$ can be taken to satisfy

$$u(x^j,t_j)\to M \ \text{ as } \ j\to\infty.$$

Meanwhile, we choose some nonnegative monotonically decreasing sequence $\{\varepsilon_j\}$ such that

$$u(x^j,t_j)=M-\varepsilon_j>0.$$

Since u vanishes in Ω^c, by regularity and without loss of generality, we may assume that $\mathrm{dist}\{x^j,\Omega^c\}\ge 1$. We construct an auxiliary function

$$v_j(x,t)=u(x,t)+\varepsilon_j\eta_j(x,t),$$

where $\eta_j(x,t)=\eta\left(\frac{x-x^j}{r},\frac{t-t_j}{r^{2s}}\right)$, r is arbitrary and fixed, and $\eta(x,t)$ satisfies

$$\eta(x,t)=\begin{cases}1, & \text{if } |x|\le\frac{1}{2},\ \ |t|\le\frac{1}{2},\\ 0, & \text{if } |x|\ge 1,\ \ |t|\ge 1.\end{cases}$$

Denote

$$Q_r(x^j,t_j):=\left\{(x,t):\left|\frac{x-x^j}{r}\right|<1,\quad \left|\frac{t-t^j}{r^{2s}}\right|<1\right\}.$$

According to the definition of $v_j(x,t)$, there holds that

$$v_j(x^j,t_j)=M>0,$$

while for $(x,t)\in(\mathbb{R}^n\times\mathbb{R})\backslash Q_r\left(x^j,t_j\right)$,

$$v_j(x,t)\le M.$$

Therefore, $v_j(x,t)$ reaches the maximum value in $Q_r\left(x^j,t_j\right)$. Set the maximum point as $\left(\overline{x}^j,\bar{t}_j\right)$. Then $v_j\left(\overline{x}^j,\bar{t}_j\right)$ satisfies

$$M+\varepsilon_j\ge v_j\left(\overline{x}^j,\bar{t}_j\right)=\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} v_j(x,t)\ge M>0. \tag{2.1}$$

Next, we also have

$$(-\Delta)_a^s v_j\left(\overline{x}^j,\bar{t}_j\right)=C_{n,s}P.V.\int_{\mathbb{R}^n}\frac{v_j\left(\overline{x}^j,\bar{t}_j\right)-v_j\left(y,\bar{t}_j\right)}{|\overline{x}^j-y|^{n+2s}}\mathrm{d}y\ge 0 \tag{2.2}$$

and

$$\frac{\partial v_j}{\partial t}\left(\overline{x}^j,\bar{t}_j\right)=0.$$

Therefore,

$$\left|\frac{\partial u}{\partial t}\left(\overline{x}^j,\bar{t}_j\right)\right|=\left|\frac{\partial v_j}{\partial t}\left(\overline{x}^j,\bar{t}_j\right)-\varepsilon_j\frac{\partial\eta_j}{\partial t}\right|\le\frac{C\varepsilon_j}{r^{2s}}, \tag{2.3}$$

which, combined with (1.7) and (2.2), implies

$$0\le(-\Delta)_a^s v_j\left(\overline{x}^j,\bar{t}_j\right)=(-\Delta)_a^s u\left(\overline{x}^j,\bar{t}_j\right)+\varepsilon_j(-\Delta)_a^s\eta_j\left(\overline{x}^j,\bar{t}_j\right)\le-\frac{\partial u}{\partial t}\left(\overline{x}^j,\bar{t}_j\right)+\frac{C\varepsilon_j}{r^{2s}}\le\frac{C\varepsilon_j}{r^{2s}}.$$

Then

$$\frac{C_0}{C_{n,s}A_1}r^{2s}(-\Delta)_a^s v_j\left(\overline{x},\bar{t}\right)\le\frac{C_0}{C_{n,s}A_1}r^{2s}\frac{C\varepsilon_j}{r^{2s}}\le C\varepsilon_j. \tag{2.4}$$

Applying (2.1) and Theorem 1.1 to v_j at $\left(\overline{x}^j,\bar{t}_j\right)$, we get

$$\frac{C_0}{C_{n,s}A_1}r^{2s}(-\Delta)_a^s v_j\left(\overline{x}^j,\bar{t}_j\right)+C_0r^{2s}\int_{B_r^c(\overline{x}^k)}\frac{v_j\left(y,\bar{t}_j\right)}{|\overline{x}^j-y|^{n+2s}}\mathrm{d}y\ge v_j\left(\overline{x}^j,\bar{t}_j\right)\ge M \tag{2.5}$$

for any $r>0$ with

$$C_0=\frac{1}{\displaystyle\int_{B_1^c(0)}\frac{1}{|y|^{n+2s}}\mathrm{d}y}.$$

We are sufficed to estimate the upper bound of the term

$$C_0r^{2s}\int_{B_r^c(\overline{x}^j)}\frac{v_j\left(y,\bar{t}_j\right)}{\left|\overline{x}^j-y\right|^{n+2s}}\mathrm{d}y$$

in (2.5). Applying (1.6), we obtain

$$\varliminf_{R\to\infty}\frac{\left|B_R(\overline{x}^j)\backslash B_{R/\sqrt[n]{2}}(\overline{x}^j)\cap\Omega^c\right|}{\left|B_R(\overline{x}^j)\right|}=\frac{1}{2}\varliminf_{R\to\infty}\frac{\left|B_R(\overline{x}^j)\cap\Omega^c\right|}{\left|B_R(\overline{x}^j)\right|}>0.$$

This means that there is a positive constant ρ and a large enough number R_j such that

$$\frac{\left|B_R(\overline{x}^j)\backslash B_{R/\sqrt[n]{2}}(\overline{x}^j)\cap\Omega^c\right|}{\left|B_R(\overline{x}^j)\right|}\geq\rho>0, \tag{2.6}$$

when $R\geq R_j$. Since r is arbitrary, by (2.1) and (2.6), we take $r=R_j/\sqrt[n]{2}$ to have

$$\begin{aligned}
&C_0r^{2s}\int_{B_r^c(\overline{x}^j)}\frac{v_j(y,\bar{t}_j)}{\left|\overline{x}^j-y\right|^{n+2s}}\mathrm{d}y\\
=&C_0\left(R_j/\sqrt[n]{2}\right)^{2s}\left(\int_{B^c_{R_j/\sqrt[n]{2}}(\overline{x}^j)\cap\Omega}\frac{v_j(y,\bar{t}_j)}{\left|\overline{x}^j-y\right|^{n+2s}}\mathrm{d}y+\int_{B^c_{R_j/\sqrt[n]{2}}(\overline{x}^j)\cap\Omega^c}\frac{M+\varepsilon_j}{\left|\overline{x}^j-y\right|^{n+2s}}\mathrm{d}y\right.\\
&\left.-\int_{B^c_{R_j/\sqrt[n]{2}}(\overline{x}^j)\cap\Omega^c}\frac{M+\varepsilon_j}{\left|\overline{x}^j-y\right|^{n+2s}}\mathrm{d}y+\int_{B^c_{R_j/\sqrt[n]{2}}(\overline{x}^j)\cap\Omega^c}\frac{v_j(y,\bar{t}_j)}{\left|\overline{x}^j-y\right|^{n+2s}}\mathrm{d}y\right)\\
\leq&C_0\left(R_j/\sqrt[n]{2}\right)^{2s}\int_{B^c_{R_j/\sqrt[n]{2}}(\overline{x}^j)}\frac{M+\varepsilon_j}{\left|\overline{x}^j-y\right|^{n+2s}}\mathrm{d}y-C_0\left(R_j/\sqrt[n]{2}\right)^{2s}\int_{B^c_{R_j/\sqrt[n]{2}}(\overline{x}^j)\cap\Omega^c}\frac{M+\varepsilon_j}{\left|\overline{x}^j-y\right|^{n+2s}}\mathrm{d}y\\
=&M+\varepsilon_j-C_0\left(R_j/\sqrt[n]{2}\right)^{2s}\int_{B^c_{R_j/\sqrt[n]{2}}(\overline{x}^j)\cap\Omega^c}\frac{M+\varepsilon_j}{\left|\overline{x}^j-y\right|^{n+2s}}\mathrm{d}y\\
\leq&M+\varepsilon_j-C_0\left(R_j/\sqrt[n]{2}\right)^{2s}\int_{(B_{R_j}(\overline{x}^j)\backslash B_{R_j/\sqrt[n]{2}}(\overline{x}^j))\cap\Omega^c}\frac{M+\varepsilon_j}{\left|\overline{x}^j-y\right|^{n+2s}}\mathrm{d}y\\
\leq&M+\varepsilon_j-C_0\left(R_j/\sqrt[n]{2}\right)^{2s}(M+\varepsilon_j)R_j^{-(n+2s)}\left|\left(B_{R_j}(\overline{x}^j)\backslash B_{R_j/\sqrt[n]{2}}(\overline{x}^j)\right)\cap\Omega^c\right|\\
\leq&M+\varepsilon_j-(M+\varepsilon_j)C_0\left(R_j/\sqrt[n]{2}\right)^{2s}R_j^{-(n+2s)}\rho|B_{R_j}(\overline{x}^j)|\\
=&(1-\theta)(M+\varepsilon_j),
\end{aligned}\tag{2.7}$$

where θ is a positive constant satisfies $0<\theta<1$. Combining (2.4) and (2.7), it holds that

$$M\leq C\varepsilon_j+(1-\theta)(M+\varepsilon_j),$$

which is obviously impossible when j is large enough and leads to a contradiction. Thus the proof of Theorem 1.2 is completed. □

2.2 A comparison principle

Lemma 2.1 (A comparison principle)**.** *Let Γ be a bounded domain in $\mathbb{R}^n$. Assume that $u,v \in C^1\left(\mathbb{R};C^{1,1}_{loc}(\Gamma)\cap\mathcal{L}_{2s}\right)$, $u-v$ is lower semi-continuous on $\overline{\Gamma}$ and satisfy*

$$\begin{cases} \frac{\partial u}{\partial t}(x,t)+(-\Delta)^s_a u(x,t) \geq \frac{\partial v}{\partial t}(x,t)+(-\Delta)^s_a v(x,t), & (x,t)\in\Gamma\times\mathbb{R}, \\ u(x,t)\geq v(x,t), & (x,t)\in\Gamma^c\times\mathbb{R}, \end{cases} \tag{2.8}$$

then $u\geq v,(x,t)\in\Gamma\times\mathbb{R}$. Futher more, if $u(x^0,t_0)=v(x^0,t_0)$ at some $(x^0,t_0)\in\Gamma\times\mathbb{R}$, then $u(x,t_0)=v(x,t_0)$ almost everywhere in $x\in\mathbb{R}^n$.

Proof. Let $w(x,t)=u(x,t)-v(x,t)$. Assume that the conclusion is not true. Since w is lower semi-continuous on $\overline{\Gamma}$, one can find a point $w(x^0,t_0)\in\Gamma\times\mathbb{R}$ such that $w(x^0,t_0)\leq \min_{\Gamma\times\mathbb{R}} w(x,t)<0$. Then by the definition of $(-\Delta)^s_a$ and the second inequality in (2.8), we have

$$\begin{aligned} &\frac{\partial w}{\partial t}(x^0,t_0)+(-\Delta)^s_a u(x^0,t_0)-(-\Delta)^s_a v(x^0,t_0) \\ =&0+C_{n,s}P.V.\int_{\mathbb{R}^n}\frac{a(x-y)\left(\left(u(x^0,t_0)-u(y,t_0)\right)-\left(v(x^0,t_0)-v(y,t_0)\right)\right)}{|x^0-y|^{n+2s}}\mathrm{d}y \\ =&C_{n,s}P.V.\int_{\mathbb{R}^n}a(x-y)\frac{w(x^0,t_0)-w(y,t_0)}{|x^0-y|^{n+2s}}\mathrm{d}y \\ \leq&C_{n,s}P.V.\int_{\Gamma^c}a(x-y)\frac{w(x^0,t_0)-w(y,t_0)}{|x^0-y|^{n+2s}}\mathrm{d}y \\ <&0. \end{aligned}$$

That is a contradiction to the first inequality in (2.8). Hence we derive $u(x,t)\geq v(x,t),(x,t)\in\Gamma\times\mathbb{R}$.

Once there exists $(x^0,t_0)\in\Gamma\times\mathbb{R}$ such that $w(x^0,t_0)=0$, then

$$\begin{aligned} &\frac{\partial w}{\partial t}(x^0,t_0)+(-\Delta)^s_a u(x^0,t_0)-(-\Delta)^s_a v(x^0,t_0) \\ =&0+C_{n,s}P.V.\int_{\mathbb{R}^n}\frac{a(x-y)\left(\left(u(x^0,t_0)-u(y,t_0)\right)-\left(v(x^0,t_0)-v(y,t_0)\right)\right)}{|x^0-y|^{n+2s}}\mathrm{d}y \\ \leq&0. \end{aligned}$$

However, the first inequality in (2.8) shows

$$\frac{\partial u}{\partial t}(x,t)+(-\Delta)^s_a u(x,t)-\frac{\partial v}{\partial t}(x,t)-(-\Delta)^s_a v(x,t)\geq 0.$$

Then according to $w(x,t)\geq 0$, we have $w(x,t_0)=0$ almost everywhere in $\mathbb{R}^n$, that is $u(x,t_0)\equiv v(x,t_0)$ in $\mathbb{R}^n$. □

3 Monotonicity

Proof of Theorem 1.3: Without loss of generality, we assume that $\mu=1$ in the conditions **(a)** and **(b)**, and keep this assumption unchanged in the rest of our paper. We first claim that $u(x,t)\leq 1$ for all $(x,t)\in\mathbb{R}^n_+\times\mathbb{R}$.

Otherwise, if $u(x,t)>1$ somewhere, the contradiction can be deduced. We denote the component of the set where $u(x,t)>1$ by Q. Let $w(x,t)=u(x,t)-1$. According to the continuity of f and condition **(a)**, we have $f(t,1)=0$ and $f(t,u(x,t))\leq 0$ for $(x,t)\in Q$, then

$$\frac{\partial w}{\partial t}(x,t)+(-\Delta)^s_a w(x,t)=\frac{\partial u}{\partial t}(x,t)+(-\Delta)^s_a u(x,t)=f(t,u(x,t))\leq 0,\quad (x,t)\in Q.$$

Since $w(x,t)\leq 0$ for $(x,t)\in Q^c$, according to the theorem 1.2, we know that $u(x,t)\leq 1$ in Q, which contradicts the assumption. So the claim is concluded. □

Proof of Theorem 1.4: For $\tau>0$ and

$$x=\left(x',x_n\right)\text{ with } x'=(x_1,\cdots,x_{n-1})\in\mathbb{R}^{n-1},$$

set

$$u_\tau(x,t)=u\left(x^\tau,t\right),\ \ x^\tau=x+\tau e_n \text{ with } e_n=\left(0',1\right),$$

and

$$w_\tau(x,t)=u(x,t)-u_\tau(x,t).$$

Our proof will be carried out in three steps. First, we show that for a τ large enough, one has

$$w_\tau(x,t)\leq 0,\quad (x,t)\in\mathbb{R}^n_+\times\mathbb{R}, \tag{3.1}$$

which provides a starting position to slide the domain. Then, we decrease τ to its limit position where (3.1) still holds, and define

$$\tau_0=\inf\{\tau|w_\tau(x,t)\leq 0\},\quad (x,t)\in\mathbb{R}^n_+\times\mathbb{R}.$$

We prove $\tau_0=0$, which means that $u(x,t)$ must be strictly monotone increasing in x_n uniformly for $t\in\mathbb{R}$.

Step 1. We prove that for a sufficiently large τ, one has

$$w_\tau(x,t)\leq 0,\quad (x,t)\in\mathbb{R}^n_+\times\mathbb{R}. \tag{3.2}$$

According to the condition on u, there is a large enough $a>0$ to make

$$u(x,t)\geq 1-\delta\ \text{ for }\ x_n\geq a,\quad (x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}. \tag{3.3}$$

From condition **(b)**, (3.3) means that f is non-increasing when $x_n \geq a$. If (3.2) is not true, then there is a positive constant M such that

$$\sup_{(x,t)\in\mathbb{R}^n_+\times\mathbb{R}} w_\tau(x,t) = M > 0. \tag{3.4}$$

Construct an auxiliary function

$$\tilde{w}_\tau(x,t) = w_\tau(x,t) - \frac{M}{2}.$$

We will apply the maximum principle in unbounded domains (Theorem 1.2) to deduce

$$\tilde{w}_\tau(x,t) \leq 0, \quad (x,t)\in\mathbb{R}^n_+\times\mathbb{R},$$

which is a contradiction.

Since $u(x,t)\to 1$ as $x_n\to+\infty$ uniformly in t, we can choose a constant $\gamma > a$ large enough so that

$$w_\tau(x,t) \leq \frac{M}{2} \ \text{ for } \ x_n \geq \gamma, \quad (x',t)\in\mathbb{R}^{n-1}\times\mathbb{R},$$

which means

$$\tilde{w}_\tau(x,t) \leq 0 \ \text{ for } \ x_n \geq \gamma, \quad (x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}. \tag{3.5}$$

Denote

$$H = \mathbb{R}^{n-1}\times(0,\gamma).$$

Then (3.5) yields

$$\tilde{w}_\tau(x,t) \leq 0, \quad (x,t)\in H^c\times\mathbb{R}. \tag{3.6}$$

Hence $\tilde{w}_\tau(x,t)$ satisfies the external condition of Theorem 1.2.

Next, we check whether $\tilde{w}_\tau(x,t)$ satisfies the condition of differential equation in $H\times\mathbb{R}$. By the definition of $\tilde{w}_\tau(x,t)$ and (1.1), we have

$$\begin{aligned}\frac{\partial\tilde{w}_\tau}{\partial t}(x,t) + (-\Delta)^s_a\tilde{w}_\tau(x,t) &= \frac{\partial w_\tau}{\partial t}(x,t) + (-\Delta)^s_a w_\tau(x,t)\\ &= f(t,u(x,t)) - f(t,u_\tau(x,t)).\end{aligned} \tag{3.7}$$

If $0 < x_n < a$, since $\tau \geq a$, then $x_n+\tau \geq a$. It can be seen from (3.3) that there is

$$u(x,t) > u_\tau(x,t) \geq 1-\delta$$

at the points where $w_\tau(x,t) > 0$, which together with condition **(b)**, the monotonicity assumption on the function f, implies that

$$f(t,u(x,t)) \leq f(t,u_\tau(x,t)), \quad 0<x_n<a, \quad (x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}, \tag{3.8}$$

at the points where $w_\tau(x,t) > 0$.

If $x_n > a$, It can be seen from (3.3) that there is

$$u(x,t) > u_\tau(x,t) \geq 1-\delta$$

at the points where $w_\tau(x,t) > 0$, therefore, by the monotonicity assumption **(b)** on the function f similarly, we have

$$f(t,u(x,t)) \leq f(t,u_\tau(x,t)), \quad x_n > a, \quad (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}, \tag{3.9}$$

at the points where $w_\tau(x,t) > 0$.

Combining (3.7)-(3.9), we have

$$\frac{\partial \tilde{w}_\tau}{\partial t}(x,t) + (-\Delta)^s_a \tilde{w}_\tau(x,t) \leq 0 \ \text{ at the points in } H \times \mathbb{R} \text{ where } w_\tau(x,t) > 0,$$

which together with (3.6) satisfies the conditions of Theorem 1.2. Thus we use the maximum principle in unbounded domains to derive

$$\tilde{w}_\tau(x,t) \leq 0, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}.$$

That is,

$$w_\tau(x,t) \leq \frac{M}{2}, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R},$$

a contradiction to the fact $\sup_{(x,t)\in\mathbb{R}^n_+ \times \mathbb{R}} w_\tau(x,t) = M > 0$ in (3.4). Therefore,

$$w_\tau(x,t) \leq 0 \ \text{ for any } \tau \geq a \ \text{ and } (x,t) \ \in \mathbb{R}^n_+ \times \mathbb{R} \tag{3.10}$$

was proved. The proof of step 1 has been completed.

Step 2. In the first step, the inequality (3.10) provides a starting position for sliding. In this present step, we decrease τ from a and prove that for any $0 < \tau < a$, we still have

$$w_\tau(x,t) \leq 0, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}. \tag{3.11}$$

In order to prove (3.11), we introduce

$$\tau_0 = \inf\{\tau | w_\tau(x,t) \leq 0\}, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R},$$

and prove $\tau_0 = 0$ by contradiction. Precisely we prove that if $\tau_0 > 0$,

$$w_\tau(x,t) \leq 0, \quad (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}, \quad \tau \in (\tau_0 - \varepsilon, \tau_0]$$

is still true after appropriately decreasing τ_0, which is violate to the definition of τ_0.

(i) We first prove that

$$\sup_{0<x_n<a,(x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}} w_{\tau_0}(x,t) < 0 \tag{3.12}$$

on $x_n \in (0,a)$, and then we can get

$$\sup_{0<x_n<a,(x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}} w_\tau(x,t) \leq 0, \quad \forall \tau \in (\tau_0-\varepsilon,\tau_0]$$

for a small constant $\varepsilon > 0$ due to the continuity of $w_\tau(x,t)$ with respect to τ.

Suppose (3.12) is false, then

$$\sup_{0<x_n<a,(x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}} w_{\tau_0}(x,t) = 0.$$

Take some

$$\{(x^j,t_j)\} \subset (\mathbb{R}^{n-1}\times[0,a])\times\mathbb{R}$$

such that

$$w_{\tau_0}(x^j,t_j) \to 0 \quad \text{as} \quad j\to\infty. \tag{3.13}$$

There exists some non-negative sequence $\{\varepsilon_j\}$ such that

$$w_{\tau_0}(x^j,t_j) = -\varepsilon_j. \tag{3.14}$$

We now prove that x^j is away from $\{x_n=0\}$ by constructing a sub-solution. Denote $\partial\Omega = \{x_n = 0\}$. First, we introduce a function $\phi(x) = (1-|x|^2)_+^s$ with

$$|(-\Delta)_a^s \phi(x)| \leq C, \quad \forall x \in B_1(0).$$

For $z \in \partial\Omega$, we set

$$r_z = \text{dist}(z+\tau_0 e_n, \partial\Omega).$$

Then for any $z\in\partial\Omega$, there exists a ball $B := B_{r_z}(z+\tau_0 e_n) \subset \Omega$. Let $\phi_{r_z}(x) = \phi(\frac{x-z-z_0e_n}{r_z})$, then

$$|(-\Delta)_a^s \phi_{r_z}(x)| \leq C, \quad \forall x \in B_{r_z}(z+z_0e_n). \tag{3.15}$$

For some fixed $R_0 > 2a > 0$, we set

$$E = \{x \in \mathbb{R}_+^n : \text{dist}(x,\partial\Omega) \geq R_0\}.$$

Then according to (3.3), $u(x,t) \geq 1-\delta$ when $x\in E$. Denote $1-\delta = \varepsilon_1$. Let $\widetilde{E} \subset\subset E$, take

$$\eta \in C_0^\infty(E), \quad 0 \leq \eta(x) \leq 1, \text{ and } \eta(x) \equiv 1 \text{ on } \tilde{E}.$$

We construct a sub-solution

$$\underline{u}(x,t) = \eta u(x,t) + \varepsilon\phi_{r_z}(x),$$

since $B\cap E = \emptyset$ when R_0 is large enough, it can be deduced that

$$\frac{\partial \underline{u}}{\partial t}(x,t) = \eta\frac{\partial u}{\partial t}(x,t) = 0, \quad (x,t)\in B\times\mathbb{R} \tag{3.16}$$

and

$$
\begin{aligned}
&(-\Delta)_a^s \underline{u}(x,t)\\
=&C_{n,s}P.V.\int_{\mathbb{R}^n}\frac{a(x-y)(\underline{u}(x,t)-\underline{u}(y,t))}{|x-y|^{n+2s}}\mathrm{d}y\\
=&C_{n,s}P.V.\int_{\mathbb{R}^n}\frac{a(x-y)(\eta u(x,t)+\varepsilon\phi_{r_z}(x)-\eta u(y,t)-\varepsilon\phi_{r_z}(y))}{|x-y|^{n+2s}}\mathrm{d}y\\
=&C_{n,s}P.V.\left(\int_{B^c}\frac{-a(x-y)\eta u(y,t)}{|x-y|^{n+2s}}\mathrm{d}y\right)\\
&\quad+C_{n,s}P.V.\left(\int_{B^c}\frac{a(x-y)\varepsilon\phi_{r_z}(x)}{|x-y|^{n+2s}}\mathrm{d}y+\int_B\frac{a(x-y)(\varepsilon\phi_{r_z}(x)-\varepsilon\phi_{r_z}(y))}{|x-y|^{n+2s}}\mathrm{d}y\right)\\
\le&\widetilde{C}\left(\int_{B^c\cap\widetilde{E}}\frac{-A_1\eta u(y,t)}{|x-y|^{n+2s}}\mathrm{d}y+\varepsilon(-\Delta)_a^s\phi_{r_z}(x)\right)\\
\le&\widetilde{C}(C\varepsilon-A_1C_{R_0}\varepsilon_1).
\end{aligned}\tag{3.17}
$$

We can choose $\varepsilon\le\varepsilon_1A_1C_{R_0}C^{-1}:=\varepsilon_0$, then (3.17) yields

$$
(-\Delta)_a^s\underline{u}(x,t)\le 0,\quad (x,t)\in B\times\mathbb{R},\tag{3.18}
$$

which combines (3.16) give that

$$
\frac{\partial\underline{u}}{\partial t}(x,t)+(-\Delta)_a^s\underline{u}(x,t)\le 0,\quad (x,t)\in B\times\mathbb{R}.
$$

In addition, we have $u(x,t)\ge\underline{u}(x,t)$ for $(x,t)\in B^c\times\mathbb{R}$. Thus according to the comparison principle (Lemma 2.1), for any $z\in\partial\Omega$, $t\in\mathbb{R}$, we have

$$
u_{\tau_0}(z,t)=u(z+\tau_0e_n,t)\ge\underline{u}(z+\tau_0e_n,t)\ge\frac{\varepsilon_0}{2}\phi_{r_z}(z+\tau_0e_n)\ge C_{\tau_0}>0,
$$

which yields

$$
w_{\tau_0}(z,t)=-u(z+\tau_0e_n,t)\le -C_{\tau_0}<0,\quad \forall z\in\partial\Omega.\tag{3.19}
$$

From (3.13) and (3.19), we conclude that x^j is away from $\partial\Omega$, which allows us to assume $B_1(x^j)\subset\mathbb{R}^n_+$ without loss of generality. Let

$$
\eta_j(x,t)=\eta\left(x-x^j,t-t_j\right)
$$

with $\eta(x,t)\in C_0^\infty(\mathbb{R}^n\times\mathbb{R})$,

$$
\eta(x,t)=\begin{cases}1, & \text{if } |x|\le\frac{1}{2},\ |t|\le\frac{1}{2},\\ 0, & \text{if } |x|\ge 1,\ |t|\ge 1,\end{cases}
$$

and construct auxiliary function

$$w_j(x,t)=w_{\tau_0}(x,t)+\varepsilon_j\eta_j(x,t). \tag{3.20}$$

Denote

$$Q'\left(x^j,t_j\right):=\left\{(x,t)\Big|\left|x-x^j\right|<1,\left|t-t^j\right|<1\right\}.$$

It is easy to prove that the perturbed function $w_j(x,t)$ takes the maximum value in $Q'\left(x^j,t_j\right)$. We will deduce a contradiction accordingly. More specifically, by (3.14), (3.20) and the definition of $w_j(x,t)$, it holds that

$$w_j\left(x^j,t_j\right)=0.$$

But for $(x,t)\in(\mathbb{R}^n_+\times\mathbb{R})\backslash Q'\left(x^j,t_j\right)$, in view of $\eta_j(x,t)=0$, one has

$$w_j(x,t)\leq 0.$$

Thus $w_j(x,t)$ attains its maximum value in $Q'\left(x^j,t_j\right)$ at a certain point denoted by $\left(\overline{x}^j,\overline{t}_j\right)$, i.e.,

$$\varepsilon_j\geq w_j\left(\overline{x}^j,\overline{t}_j\right)=\sup_{(x,t)\in\mathbb{R}^n_+\times\mathbb{R}} w_j(x,t)\geq 0. \tag{3.21}$$

Then we introduce an auxiliary function of translation

$$\overline{w}_j(x,t)=w_j\left(x+\overline{x}^j,t+\overline{t}_j\right),$$

so that (3.21) can be written as

$$\varepsilon_j\geq\overline{w}_j(0,0)=\sup_{(x,t)\in\mathbb{R}^n_+\times\mathbb{R}}\overline{w}_j(x,t)\geq 0. \tag{3.22}$$

Then we have

$$(-\Delta)^s_a\overline{w}_j(0,0)=C_{n,s}P.V.\int_{\mathbb{R}^n}\frac{a(x-y)\left(\overline{w}_j(0,0)-\overline{w}_j(y,0)\right)}{|y|^{n+2s}}\mathrm{d}y\geq 0 \tag{3.23}$$

and

$$\frac{\partial\overline{w}_j}{\partial t}(0,0)=0. \tag{3.24}$$

Therefore, from

$$w_{\tau_0}(\overline{x}^j,\overline{t}_j)\to 0,\quad \text{as } j\to\infty,$$

which is deduced by the definition of $w_j(x,t)$ in (3.20) and (3.21), we get

$$\begin{aligned}
0&\geq(-\Delta)^s_a\overline{w}_j(0,0)\\
&=(-\Delta)^s_a w_{\tau_0}\left(\overline{x}^j,\overline{t}_j\right)+\varepsilon_j(-\Delta)^s_a\eta_j\left(\overline{x}^j,\overline{t}_j\right)\\
&=-\frac{\partial w_{\tau_0}}{\partial t}\left(\overline{x}^j,\overline{t}_j\right)+f\left(\overline{t}_j,u\left(\overline{x}^j,\overline{t}_j\right)\right)-f\left(\overline{t}_j,u_{\tau_0}\left(\overline{x}^j,\overline{t}_j\right)\right)+\varepsilon_j(-\Delta)^s_a\eta_j\left(\overline{x}^j,\overline{t}_j\right)
\end{aligned}$$

$$
\begin{aligned}
&=-\frac{\partial \overline{w}_j}{\partial t}(0,0)+\varepsilon_j\frac{\partial \eta_j}{\partial t}\left(\overline{x}^j,\overline{t}_j\right)+f\left(\overline{t}_j,u\left(\overline{x}^j,\overline{t}_j\right)\right)-f\left(\overline{t}_j,u_{\tau_0}\left(\overline{x}^j,\overline{t}_j\right)\right)+\varepsilon_j(-\Delta)_a^s\eta_j\left(\overline{x}^j,\overline{t}_j\right)\\
&\le C\varepsilon_j+f\left(\overline{t}_j,u\left(\overline{x}^j,\overline{t}_j\right)\right)-f\left(\overline{t}_j,u_{\tau_0}\left(\overline{x}^j,\overline{t}_j\right)\right)\to 0, \quad \text{as } j\to\infty.
\end{aligned}\tag{3.25}
$$

In addition, the generalized weighted average inequality (Theorem 1.1) implies

$$
\frac{C_0}{C_{n,s}A_1}r^{2s}(-\Delta)_a^s\overline{w}_j(0,0)+C_0r^{2s}\int_{B_r^c(0)}\frac{\overline{w}_j(y,0)}{|y|^{n+2s}}\mathrm{d}y\ge \overline{w}_j(0,0) \quad \text{for any } r>0 \tag{3.26}
$$

with

$$
C_0=\frac{1}{\displaystyle\int_{B_r^c(0)}\frac{1}{|y|^{n+2s}}\mathrm{d}y},
$$

which together with (3.22) and (3.25) yields that for any finite $r>0$,

$$
\int_{B_r^c(0)}\frac{\overline{w}_j(y,0)}{|y|^{n+2s}}\mathrm{d}y\to 0, \quad \text{as } j\to\infty.
$$

Since

$$
\overline{w}_j(y,0)=w_j\left(y+\overline{x}^j,\overline{t}_j\right)=w_{\tau_0}\left(y+\overline{x}^j,\overline{t}_j\right)+\varepsilon_j\eta_j\left(y+\overline{x}^j,\overline{t}_j\right),
$$

by the definition of η_j, we know that there is a constant $\overline{r}>0$ such that $\eta_j=0$ and $\overline{w}_j(y,0)=w_{\tau_0}\left(y+\overline{x}^j,\overline{t}_j\right)\le 0$ for any $|y|>\overline{r}$. Furthermore, for any fixed $r\ge\overline{r}>0$, we have

$$
\overline{w}_j(y,0)\to 0, \quad y\in B_r^c(0), \quad \text{as } j\to\infty. \tag{3.27}
$$

Since $u(x,t)$ is uniformly continuous, the Arzelà-Ascoli theorem can be used to show that $u_j(x,t):=u\left(x+\overline{x}^j,t+\overline{t}_j\right)$ has a convergent sub-sequence (still denote by itself), which satisfies

$$
u_j(x,t)\to u_\infty(x,t), \quad (x,t)\in\mathbb{R}_+^n\times\mathbb{R}, \quad \text{as } j\to\infty.
$$

Combining the above formula and (3.27), we obtain

$$
u_\infty(x,0)-(u_\infty)_{\tau_0}(x,0)\equiv 0, \quad x\in B_r^c(0).
$$

Hence for any $k\in\mathbb{N}$ and any fixed $r\ge\overline{r}>0$, we have

$$
\begin{aligned}
u_\infty\left(x',x_n,0\right)&=u_\infty\left(x',x_n+\tau_0,0\right)=u_\infty\left(x',x_n+2\tau_0,0\right)\\
&=\cdots=u_\infty\left(x',x_n+k\tau_0,0\right), \quad x\in B_r^c(0).
\end{aligned}\tag{3.28}
$$

However, from the asymptotic assumption that

$$
u_\infty\left(x',x_n,0\right)\xrightarrow{x_n\to+\infty}1 \ \text{ uniformly in } x'=(x_1,\cdots,x_{n-1})
$$

and

$$
u_\infty\left(x',x_n,0\right)=0 \quad \text{for} \quad x_n\le 0,
$$

we obtain $u_\infty(x',x_n,0)=0$ with $x_n\leq 0$ and $u_\infty(x',x_n+k\tau_0,0)\to 1$ when taking k sufficiently large , which contradicts (3.28). Hence (3.12) holds.

(ii) If $\tau_0>0$, to deduce a contradiction, we are to prove that there exists a constant $\varepsilon>0$ such that

$$w_\tau(x,t)\leq 0,\quad (x,t)\in\mathbb{R}^n_+\times\mathbb{R},\quad \forall\tau\in(\tau_0-\varepsilon,\tau_0]. \tag{3.29}$$

According to the continuity of $w_\tau(x,t)$ with respect to τ, (3.12) yields that there exists a small constant $\varepsilon>0$ such that

$$\sup_{0<x_n<a,(x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}} w_\tau(x,t)\leq 0,\quad \tau\in(\tau_0-\varepsilon,\tau_0]. \tag{3.30}$$

Therefore, we only need to show

$$\sup_{x_n>a,(x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}} w_\tau(x,t)\leq 0,\quad \tau\in(\tau_0-\varepsilon,\tau_0]. \tag{3.31}$$

We verify (3.29) by contradiction. Suppose (3.31) is not true, then there exists some $\tau\in(\tau_0-\varepsilon,\tau_0]$ and a constant $M>0$ such that

$$\sup_{x_n>a,(x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}} w_\tau(x,t):=M>0. \tag{3.32}$$

Similar to the first step, we construct an auxiliary function

$$v_\tau(x,t):=w_\tau(x,t)-\frac{M}{2}.$$

In view of the asymptotic condition **(b)** on u, a sufficiently large constant $\gamma>a$ can be found such that

$$w_\tau(x,t)\leq\frac{M}{2},\quad x_n\geq\gamma,\quad (x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}. \tag{3.33}$$

Let

$$G=\left\{(x',x_n,t)\in\mathbb{R}^{n-1}\times\mathbb{R}^n_+\times\mathbb{R}\middle|a<x_n<M,\quad (x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}\right\}.$$

Then combining (3.30), (3.33) and the assumption that $u(x,t)=0$ for $(x,t)\in\mathbb{R}^n\backslash\mathbb{R}^n_+\times\mathbb{R}$, we obtain

$$v_\tau(x,t)\leq 0,\quad (x,t)\in G^c\times\mathbb{R}, \tag{3.34}$$

which means that the exterior condition of the maximum principle in unbounded domains (Theorem 1.2) is satisfied.

Next we consider the differential inequality in G satisfied by $v_\tau(x,t)$. Since for any $\tau\in(\tau_0-\varepsilon,\tau_0]$, $(x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}$, we have

$$u(x,t)>u_\tau(x,t)+\frac{M}{2}\geq u_\tau(x,t)\geq 1-\delta$$

at the points in $G\times\mathbb{R}$ where $v_\tau(x,t)>0$ when $a<x_n<M$, then there is

$$f(t,u(x,t))-f(t,u_\tau(x,t))\le 0,\quad a<x_n<M,\quad (x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}$$

due to the monotonicity of $f(x,u)$ (condition **(b)**). Hence by using (1.1) at the points in $G\times\mathbb{R}$ where $v_\tau(x,t)>0$, we obtain

$$\begin{aligned}\frac{\partial v_\tau}{\partial t}(x,t)+(-\Delta)_a^s v_\tau(x,t)=&\frac{\partial w_\tau}{\partial t}(x,t)+(-\Delta)_a^s w_\tau(x,t)\\ =&f(t,u(x,t))-f(t,u_\tau(x,t))\le 0,\end{aligned}\tag{3.35}$$

which combines with (3.34) and the maximum principle in unbounded domains (Theorem 1.2) imply that

$$v_\tau(x,t)\le 0,\quad (x,t)\in\mathbb{R}^n_+\times\mathbb{R},\ \forall\tau\in(\tau_0-\varepsilon,\tau_0].$$

That is

$$w_\tau(x,t)\le\frac{M}{2},\quad (x,t)\in\mathbb{R}^n_+\times\mathbb{R},\ \forall\tau\in(\tau_0-\varepsilon,\tau_0],$$

which is a contradiction with the assumption (3.32). Hence, (3.31) is true and (3.29) has been proved. We derive (3.11) and the proof of Step 2 is completed.

Step 3. In this step, we prove that $u(x,t)$ is strictly increasing along the x_n-direction. From Step 1 and Step 2, we obtain

$$w_\tau(x,t)\le 0,\quad (x,t)\in\mathbb{R}^n_+\times\mathbb{R},\ \forall\tau>0.$$

To show that $u(x,t)$ is strictly increasing along the x_n-direction, we are sufficed to prove

$$w_\tau(x,t)<0,\quad (x,t)\in\mathbb{R}^n_+\times\mathbb{R},\ \forall\tau>0.\tag{3.36}$$

Otherwise, if (3.36) is not valid, then there exists a point $(x^0,t_0)\in\mathbb{R}^n_+\times\mathbb{R}$ and $\tau_0>0$ such that

$$w_{\tau_0}(x^0,t_0)=0,\tag{3.37}$$

that is, $w_{\tau_0}(x,t)$ attains the maximum value at (x^0,t_0) in $\mathbb{R}^n_+\times\mathbb{R}$ and

$$\frac{\partial w_{\tau_0}}{\partial t}(x^0,t_0)=0.\tag{3.38}$$

Then, by the definition of the fractional Laplacian and (1.1), we have

$$\begin{aligned}0=&f(t_0,u(x^0,t_0))-f(t_0,u_{\tau_0}(x^0,t_0))=\frac{\partial w_{\tau_0}}{\partial t}(x^0,t_0)+(-\Delta)_a^s w_{\tau_0}(x^0,t_0)\\ =&C_{n,s}P.V.\int_{\mathbb{R}^n}\frac{-a(x-y)w_{\tau_0}(y,t_0)}{|x^0-y|^{n+2s}}\mathrm{d}y.\end{aligned}$$

Since $a(x) \geq A_1 > 0$, it follows that

$$w_{\tau_0}(y,t_0) \equiv 0, \quad y \in \mathbb{R}^n,$$

that is, for any $k \in \mathbb{N}$, we have

$$u\left(y',y_n,t_0\right) = u\left(y',y_n+\tau_0,t_0\right) = u\left(y',y_n+2\tau_0,t_0\right) = \cdots = u\left(y',y_n+k\tau_0,t_0\right), \quad x \in \mathbb{R}^n. \quad (3.39)$$

However, by the asymptotic assumption of $u(x,t)$, we also have

$$u_\infty(x',x_n,0) \xrightarrow{x_n \to +\infty} 1 \text{ uniformly in } x' = (x_1,\cdots,x_{n-1})$$

and

$$u_\infty\left(x',x_n,0\right) = 0, \quad \text{for } x_n \leq 0.$$

Hence, (3.39) gives a contradiction and we obtain (3.36), which implies that $u(x,t)$ is strictly increasing along the x_n-direction. □

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