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Spectral Properties of the Sub-Laplacian on 2-Step Stratified Lie Groups Without the Moore-Wolf Condition

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Abstract. In this paper, we consider the structure of the L^2-spectrum of the sub-Laplacian on 2-step stratified Lie groups by using the theory of unitary irreducible representations and Hermit functions. We extend the results for Heisenberg group and H-type Lie group to more general 2-step stratified Lie groups without the Moore-Wolf condition.

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1 Introduction

Harmonic analysis on nilpotent Lie groups is by now classical matter that goes back to the first half of the 20th century (see e.g. [1] for a self-contained presentation). In fact, as is known to all, harmonic analysis on nilpotent Lie groups plays a power role in contemporary investigations of linear PDEs. Since Folland and Stein's work in [2], some subjects concerning invariant operators on nilpotent Lie groups have been paid more and more attention. Many important results have been obtained, see [3–7] and their references.

In the Euclidean case $\mathbb{R}^n$, the Fourier transform of an integrable function f may thus be seen as the function on $(\mathbb{R}^n)^*$ (usually identified to $\mathbb{R}^n$) given by

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$$\mathcal{F}(f)(\xi)=\widehat{f}(\xi)=\int_{\mathbb{R}^n}\mathrm{e}^{-i\xi\cdot x}f(x)\mathrm{d}x.$$

The Fourier transform on Euclidean has a number of noteworthy properties, we first recall that it changes convolution products into products of functions, namely

$$\mathcal{F}(f*g)=\mathcal{F}f\cdot\mathcal{F}g,\quad \forall f,g\in L^1(G).$$

What's more, we have the inverse Fourier transform

$$\forall x\in\mathbb{R}^n,\quad f(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}\mathrm{e}^{i\xi\cdot x}\widehat{f}(\xi)\mathrm{d}\xi,$$

and the Fourier-Plancherel formulae identity

$$\int_{\mathbb{R}^n}|f(x)|^2\mathrm{d}x=\int_{\mathbb{R}^n}|\widehat{f}(\xi)|^2\mathrm{d}\xi.$$

Among the numerous additional properties of the Fourier transform on $\mathbb{R}^n$, let us just underline that it allows to "diagonalize" the Laplace operator, namely for all smooth compactly supported functions, we have

$$\mathcal{F}(\Delta f)(\xi)=-|\xi|^2\widehat{f}(\xi). \tag{1.1}$$

For noncommutative groups, the simplest example (apart from $\mathbb{R}^n$) of a nilpotent Lie group is the Heisenberg group, and the harmonic analysis there is a very well researched topic. We do not intend to make an overview of the subject here, but we refer to the books of Stein [8] and Thangavelu [9] for an introduction to the harmonic analysis on the Heisenberg group and for the historic development of the area. If we consider the Heisenberg group $\mathbb{H}^d=\mathbb{R}^{2d+1}$ whose elements $w\in\mathbb{R}^{2d+1}$ can be written $w=(x,y,s)$ with $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$, endowed with the following product law:

$$w\cdot w'=(x,y,s)\cdot(x',y',s')=(x+x',y+y',s+s'-2x\cdot y'+2y\cdot x')$$

where for $x,x'\in\mathbb{R}^d$, $x\cdot x'$ denotes the Euclidean scalar product of the vectors x and x'. Equipped with the standard differential structure of the manifold $\mathbb{R}^{2d+1}$, the set $\mathbb{H}^d$ is a noncommutative Lie group with identity $(0,0)$.

Let us next recall the definition of the sub-Laplacian on the Heisenberg group, that will play a fundamental role as the Laplacian in $\mathbb{R}^n$. Being a real Lie group, the Heisenberg group may be equipped with a linear space of left invariant vector fields, that is vector fields commuting with any left translation $\tau_w(w')=w\cdot w'$. It is well known that this linear space has dimension $2d+1$ and is generated by the vector fields

$$S=\partial_s,\quad X_j=\partial_{x_j}+2y_j\partial_s\quad\text{and}\quad Y_j=\partial_{y_j}-2x_j\partial_s,\quad 1\leqslant j\leqslant d.$$