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Spectral Properties of the Sub-Laplacian on 2-Step Stratified Lie Groups Without the Moore-Wolf Condition

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Abstract. In this paper, we consider the structure of the L^2-spectrum of the sub-Laplacian on 2-step stratified Lie groups by using the theory of unitary irreducible representations and Hermit functions. We extend the results for Heisenberg group and H-type Lie group to more general 2-step stratified Lie groups without the Moore-Wolf condition.

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1 Introduction

Harmonic analysis on nilpotent Lie groups is by now classical matter that goes back to the first half of the 20th century (see e.g. [1] for a self-contained presentation). In fact, as is known to all, harmonic analysis on nilpotent Lie groups plays a power role in contemporary investigations of linear PDEs. Since Folland and Stein's work in [2], some subjects concerning invariant operators on nilpotent Lie groups have been paid more and more attention. Many important results have been obtained, see [3–7] and their references.

In the Euclidean case $\mathbb{R}^n$, the Fourier transform of an integrable function f may thus be seen as the function on $(\mathbb{R}^n)^*$ (usually identified to $\mathbb{R}^n$) given by

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$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-i\xi\cdot x} f(x)\mathrm{d}x.$$

The Fourier transform on Euclidean has a number of noteworthy properties, we first recall that it changes convolution products into products of functions, namely

$$\mathcal{F}(f*g) = \mathcal{F}f\cdot\mathcal{F}g, \quad \forall f,g\in L^1(G).$$

What's more, we have the inverse Fourier transform

$$\forall x\in\mathbb{R}^n, \quad f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot x}\widehat{f}(\xi)\mathrm{d}\xi,$$

and the Fourier-Plancherel formulae identity

$$\int_{\mathbb{R}^n} |f(x)|^2\mathrm{d}x = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2\mathrm{d}\xi.$$

Among the numerous additional properties of the Fourier transform on $\mathbb{R}^n$, let us just underline that it allows to "diagonalize" the Laplace operator, namely for all smooth compactly supported functions, we have

$$\mathcal{F}(\Delta f)(\xi) = -|\xi|^2\widehat{f}(\xi). \tag{1.1}$$

For noncommutative groups, the simplest example (apart from $\mathbb{R}^n$) of a nilpotent Lie group is the Heisenberg group, and the harmonic analysis there is a very well researched topic. We do not intend to make an overview of the subject here, but we refer to the books of Stein [8] and Thangavelu [9] for an introduction to the harmonic analysis on the Heisenberg group and for the historic development of the area. If we consider the Heisenberg group $\mathbb{H}^d = \mathbb{R}^{2d+1}$ whose elements $w\in\mathbb{R}^{2d+1}$ can be written $w = (x,y,s)$ with $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$, endowed with the following product law:

$$w\cdot w' = (x,y,s)\cdot(x',y',s') = (x+x',y+y',s+s'-2x\cdot y'+2y\cdot x')$$

where for $x,x'\in\mathbb{R}^d$, $x\cdot x'$ denotes the Euclidean scalar product of the vectors x and x'. Equipped with the standard differential structure of the manifold $\mathbb{R}^{2d+1}$, the set $\mathbb{H}^d$ is a noncommutative Lie group with identity $(0,0)$.

Let us next recall the definition of the sub-Laplacian on the Heisenberg group, that will play a fundamental role as the Laplacian in $\mathbb{R}^n$. Being a real Lie group, the Heisenberg group may be equipped with a linear space of left invariant vector fields, that is vector fields commuting with any left translation $\tau_w(w') = w\cdot w'$. It is well known that this linear space has dimension $2d+1$ and is generated by the vector fields

$$S = \partial_s, \quad X_j = \partial_{x_j} + 2y_j\partial_s \quad \text{and} \quad Y_j = \partial_{y_j} - 2x_j\partial_s, \quad 1\leqslant j\leqslant d.$$

The sub-Laplacian associated to the vector fields $\left(X_j\right)_{1\leqslant j\leqslant d}$ and $\left(Y_j\right)_{1\leqslant j\leqslant d}$ reads

$$\Delta_{\mathbb{H}}\stackrel{\text{def}}{=}\sum_{j=1}^{d}\left(X_j^2+Y_j^2\right).$$

As in the Euclidean case, the Fourier transform allows to diagonalize operator $\Delta_{\mathbb{H}}$, a property that is based on the following relation that holds true for all functions f and u in $\mathcal{S}\left(\mathbb{H}^d\right)$ and $\mathcal{S}\left(\mathbb{R}^d\right)$:

$$\mathcal{F}_{\mathbb{H}}\left(\Delta_{\mathbb{H}}f\right)(\lambda)=4\mathcal{F}_{\mathbb{H}}(f)(\lambda)\circ\Delta_{\mathrm{osc}}^{\lambda}$$

with

$$\Delta_{\mathrm{osc}}^{\lambda}u(x)\stackrel{\text{def}}{=}\sum_{j=1}^{d}\partial_j^2u(x)-\lambda^2|x|^2u(x).$$

This prompts us to take advantage of the spectral structure of the harmonic oscillator to get an analog of formula (1.1). In particular, by harmonic analysis on the harmonic oscillator, we can compute the explicit formula of the heat kernel and fundamental solution, see [10–15].

In another aspect, the Laplacian on complete Riemannian manifolds is essentially self-adjoint on the space of smooth functions with compact supports and so it is interesting to study the structure of the spectrum of Laplacians on such manifolds ([16, 17]). For compact Riemannian manifolds one of the most interesting problems relating to the spectrum is the isospectral problem, see [18–21].

For non-compact manifolds such a problem is not so interesting from the same view point as compact manifolds and the problems which concerned are the existence or the non-existence of L^2-eigenvalues, the size of the continuous spectrum and characterizations of the lowest bound of the spectral set in terms of the geometrical data, or the explicit determination of the spectral set. In this case, complete Riemannian manifolds of non-negative curvature with some additional conditions, similar results are proved in [16, 17]. For a 2-step nilpotent Lie group $\mathbb{G}$ with certain conditions (which are satisfied by the Heisenberg group, for example), Furutani et.al. [22] introduce a Fourier transformation on $\mathbb{G}$ by means of the representations of $\mathbb{G}$ based on the Kirillov theory, and analyse the sub-Laplacian on the space of transformed functions, where the sub-Laplacian takes a simpler form. They prove that the spectral set $\sigma(\Delta_{\mathbb{H}})$ of the sub-Laplacian on $\mathbb{G}$ is $[0,\infty)$. Moreover, $\sigma(\Delta_{\mathbb{H}})$ consists of continuous spectrum if $\mathbb{G}$ is the Heisenberg group. Later the same results are obtained by Dasgupta et. al. [23] via different methods.

In this paper, we revise and generalize the results in [22, 23] to the 2-step stratified Lie group, where we can give explicitly all irreducible unitary representations, and we left the general nilpotent Lie group as one open problem. Then we state our main results as follows.

Theorem 1.1. *The sub-Laplacian on 2-step stratified Lie group has no point spectrum, and that the spectrum is equal to* $[0,\infty)$.

2 Fourier analysis on 2-step stratified Lie groups

In this section we recall some basics of harmonic analysis on 2-step nilpotent (stratified) Lie groups to make the paper self contained. A complete account of representation theory for connected, simply connected nilpotent Lie groups can be found in [3, 24–26].

2.1 Basic facts on nilpotent Lie groups

A Lie group $\mathbb{G}$ with Lie algebra $\mathfrak{g}$ is called nilpotent, if its lower central series

$$\mathbb{G}=\mathbb{G}_1 \rhd \mathbb{G}_2 \rhd \mathbb{G}_3 \rhd \cdots \quad \text{with } \mathbb{G}_{j+1}=\left[\mathbb{G},\mathbb{G}_j\right]$$

determines to the trivial group in finitely many steps. Here the bracket $\left[\mathbb{G},\mathbb{G}_j\right]$ denotes the commutator group, i.e. the group generated by all commutators of elements of $\mathbb{G}$ and $\mathbb{G}_j$. This condition is equivalent to the condition that the lower central series of the Lie algebra

$$\mathfrak{g}=\mathfrak{g}_1 \geq \mathfrak{g}_2 \geq \mathfrak{g}_3 \geq \cdots \quad \text{with } \mathfrak{g}_{j+1}=\left[\mathfrak{g},\mathfrak{g}_j\right]$$

determines in finitely many steps to the null-space. Here the bracket $\left[\mathfrak{g},\mathfrak{g}_j\right]$ denotes the linear subspace of $\mathfrak{g}$ generated by all brackets of elements of $\mathfrak{g}$ and $\mathfrak{g}_j$. In both cases, group and algebra definition, the minimal number of steps in the lower central series needed to arrive at the trivial group or at the null-space, respectively, is the same, say q. It is called the step of nilpotency of $\mathbb{G}$ and $\mathfrak{g}$.

Our main concern is for a special class of nilpotent Lie groups, the stratified Lie groups. Their advantage is, that they additionally admit a family of self-similarities which are automorphisms. Further these self-similarities have nice properties concerning left-invariant (sub-)Riemannian metrics on the group.

Definition 2.1. *A stratified Lie group of step q is a simply connected q-step nilpotent Lie group $\mathbb{G}$ with Lie algebra $\mathfrak{g}$ together with a grading*

$$\mathfrak{g}=V_1\oplus V_2\oplus\cdots\oplus V_d$$

with $\left[V_1,V_j\right]=V_{1+j}$ *where* $V_m=0$ *if* $m>q$.

Remark 2.1. For example, every simply connected 2-step nilpotent Lie group $\mathbb{G}$ is such a stratified nilpotent Lie group with grading $\mathfrak{g}=V_1\oplus[\mathfrak{g},\mathfrak{g}]$, where V_1 is isomorphic to $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$.

It is well-known ([27]) that for such groups, the exponential map

$$\exp:\mathfrak{g}\to\mathbb{G},$$

is a global diffeomorphism from $\mathfrak{g}$ onto $\mathbb{G}$. This map becomes a Lie isomorphism once one endows the Lie algebra $\mathfrak{g}$ with the group law given by the Baker-Campbell-Hausdorff formula, which is a polynomial map which terminates after a finite number of terms since $\mathbb{G}$ is nilpotent.

2.2 Irreducible unitary representations

In the following, we will restrict our attention to stratified Lie group of step two, which means the left-invariant Lie algebra $\mathfrak{g}$ is endowed with a vector space decomposition

$$\mathfrak{g}=\mathfrak{v}\oplus\mathfrak{z},$$

with $\dim\mathfrak{v}=n,\dim\mathfrak{z}=m$ and

$$[\mathfrak{g},\mathfrak{g}]=\mathfrak{z}\subseteq \text{the center of } \mathfrak{g}.$$

Then, there exists a decomposition $\mathbb{R}^N=\mathbb{R}^n\oplus\mathbb{R}^m$ and a bilinear, antisymmetric map

$$\sigma:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^m$$

such that, for $Z,Z'\in\mathbb{R}^n$ and $t,t'\in\mathbb{R}^m$,

$$[(Z,t),(Z',t')]=(0,\sigma(Z,Z'))$$

and

$$(Z,t)\cdot(Z',t')=\left(Z+Z',t+t'+\frac{1}{2}\sigma(Z,Z')\right). \tag{2.1}$$

The map σ and the integers n,m are determined by the group law and dimension. Conversely, for any integers n,m and any bilinear, antisymmetric map $\sigma:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^m$, one may define a Lie group of step two by the formula (2.1). We choose an inner product on $\mathfrak{g}$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal. Fix a Jacobian basis $\mathcal{B}=\{X_1,X_2\cdots,X_n,X_{n+1},\cdots,X_{n+m}\}$ so that $\mathfrak{v}=\text{span}_{\mathbb{R}}\{X_1,\cdots,X_n\}$ and $\mathfrak{z}=\text{span}_{\mathbb{R}}\{X_1,\cdots,X_m\}$. Since $\mathfrak{g}$ is nilpotent the exponential map is an analytic diffeomorphism. We can identify $\mathbb{G}$ with $\mathfrak{v}\oplus\mathfrak{z}$ and write $(X+T)$ for $\exp(X+T)$ and denote it by (X,T) where $X\in\mathfrak{v}$ and $T\in\mathfrak{z}$. The product law on $\mathbb{G}$ is given by (2.1).

Now, given $\lambda\in\mathbb{R}^m$ (the dual of $\mathfrak{z}$), we define the matrix $B^{(\lambda)}\in\mathcal{M}_n(\mathbb{R})$ as follows. For any $Z,Z'\in\mathbb{R}^n$, there holds

$$\langle\lambda,\sigma(Z,Z')\rangle=\langle Z,B^{(\lambda)}\cdot Z'\rangle.$$

If $(\iota_1,\cdots,\iota_m)$ denotes an orthonormal basis of $\mathbb{R}^m$, we also define $B_k\in\mathcal{M}_n(\mathbb{R})$ by

$$\langle\iota_k,\sigma(Z,Z')\rangle=\langle Z,B_k\cdot Z'\rangle.$$

Then for $\lambda=\sum\limits_{k=1}^m\lambda_k\iota_k$, we get

$$B^{(\lambda)}=\sum_{k=1}^m\lambda_kB_k.$$

Conversely, the map σ may be defined from $(B_k)_{1\leq k\leq m}$ thanks to the equality

$$\sigma(Z,Z')=(\langle Z,B_k\cdot Z'\rangle)_{1\leq k\leq m}.$$

Notice that the map $\lambda \mapsto B^{(\lambda)}$ is linear, with its image spanned by $(B_k)_{1\le k\le m}$. As $B^{(\lambda)}$ is an antisymmetric matrix, its rank is an even number. We define the integer d by

$$2d := \max_{\lambda\in\mathbb{R}^m} \operatorname{rank} B^{(\lambda)}.$$

The set

$$\Lambda := \left\{\lambda\in\mathbb{R}^m \,|\, \operatorname{rank} B^{(\lambda)} = 2d\right\}$$

is then a nonempty Zariski-open subset of $\mathbb{R}^m$. We denote by k the dimension of the radical r_λ of $B^{(\lambda)}$. Following [28] and [29], if $r_\lambda = \{0\}$ for each $\lambda\in\Lambda$, then the Lie algebra is called an Moore-Wolf algebras and the corresponding Lie group is called an Moore-Wolf group (MW in short). In this paper, we will only consider $\mathbb{G}$ to be a 2-step stratified Lie group without MW-condition.
For

$$(X,T) = \exp\left(\sum_{j=1}^{n} x_j X_j + \sum_{j=1}^{m} t_j X_{n+j}\right), \quad x_j, t_j \in \mathbb{R},$$

the map

$$(x_1,\cdots,x_n,t_1\cdots,t_m) \longrightarrow \sum_{j=1}^{n} x_j X_j + \sum_{j=1}^{m} t_j X_{n+j} \longrightarrow \exp\left(\sum_{j=1}^{n} x_j X_j + \sum_{j=1}^{m} t_j X_{n+j}\right)$$

takes Lebesgue measure $\mathrm{d}x_1\cdots\mathrm{d}x_n, \mathrm{d}t_1\cdots\mathrm{d}t_m$ of $\mathbb{R}^{n+m}$ to Haar measure on $\mathbb{G}$. Any measurable function f on $\mathbb{G}$ will be identified with a function on $\mathbb{R}^{n+m}$.

Therefore, there exists an orthonormal basis

$$(X_1(\lambda),\cdots,X_d(\lambda),Y_1(\lambda),\cdots,Y_d(\lambda),R_1(\lambda),\cdots,R_k(\lambda))$$

and d continuous functions

$$\eta_j : \mathbb{R}^m \to \mathbb{R}_+, \quad 1\le j\le d$$

such that $B^{(\lambda)}$ reduces to the form

$$\begin{pmatrix} 0 & \eta(\lambda) & 0 \\ -\eta(\lambda) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R})$$

where

$$\eta(\lambda) := \operatorname{diag}(\eta_1(\lambda),\cdots,\eta_d(\lambda)) \in \mathcal{M}_d(\mathbb{R})$$

and each $\eta_j(\lambda) > 0$ is smooth and homogeneous of degree 1 in $\lambda = (\lambda_1,\cdots,\lambda_m)$ and the basis vectors are chosen to depend smoothly on λ in Λ. Decomposing $\mathfrak{v}$ as

$$\mathfrak{v} = \mathfrak{p}_\lambda \oplus \mathfrak{q}_\lambda \oplus \mathfrak{r}_\lambda$$

with

$$\begin{aligned}\mathfrak{p}_\lambda &:= \mathrm{span}_{\mathbb{R}}\left(X_1(\lambda),\cdots,X_d(\lambda)\right),\\ \mathfrak{q}_\lambda &:= \mathrm{span}_{\mathbb{R}}\left(Y_1(\lambda),\cdots,Y_d(\lambda)\right),\\ \mathfrak{r}_\lambda &:= \mathrm{span}_{\mathbb{R}}\left(R_1(\lambda),\cdots,R_k(\lambda)\right).\end{aligned}$$

Then we have the decomposition $\mathfrak{g}=\mathfrak{p}_\lambda\oplus\mathfrak{q}_\lambda\oplus\mathfrak{r}_\lambda\oplus\mathfrak{z}$. We denote the element $\exp(X+Y+R+T)$ of $\mathbb{G}$ by (X,Y,R,T) for $X\in\mathfrak{p}_\lambda, Y\in\mathfrak{q}_\lambda, R\in\mathfrak{r}_\lambda, T\in\mathfrak{z}$. Further we can write

$$(X,Y,R,T)=\sum_{j=1}^{d}x_j(\lambda)X_j(\lambda)+\sum_{j=1}^{d}y_j(\lambda)Y_j(\lambda)+\sum_{j=1}^{k}r_j(\lambda)R_j(\lambda)+\sum_{j=1}^{m}t_jT_j$$

and denote it by (x,y,r,t) suppressing the dependence of λ which will be understood from the context.

As we have done in [30], for (λ,ν,w) in $\Lambda\times\mathbb{R}^k\times\mathbb{R}^N$ with

$$w=(x,y,r,t)\in\mathbb{R}^d\oplus\mathbb{R}^d\oplus\mathbb{R}^k\oplus\mathbb{R}^m=\mathbb{R}^N,$$

we define the irreducible unitary representations of $\mathbb{R}^N$, equipped with the group law of the nilpotent group defined above, on $L^2\left(\mathbb{R}^d\right)$

$$\begin{aligned}\left(\pi_{\lambda,\nu}(w)\phi\right)(\xi)&:=\exp\left(i\sum_{j=1}^{m}\lambda_jt_j+i\sum_{j=1}^{k}\nu_jr_j+i\sum_{j=1}^{d}\eta_j(\lambda)\left(y_j\xi_j+\frac{1}{2}x_jy_j\right)\right)\phi(\xi+x)\\ &=e^{i\langle\nu,r\rangle}e^{i\langle\lambda,t\rangle}e^{i\sum_{j=1}^{d}\eta_j(\lambda)\left(y_j\xi_j+\frac{1}{2}x_jy_j\right)}\phi(\xi+x)\\ &=e^{i\langle\nu,r\rangle}e^{i\langle\lambda,t\rangle}e^{i\langle\eta(\lambda)\cdot(\xi+\frac{1}{2}x),y\rangle}\phi(\xi+x).\end{aligned}$$

2.3 Some examples

The aim of this subsection is to collect some explicit examples of stratified Lie groups of step two. To begin with, we present the most studied (and by far one of the most important) among stratified Lie groups, the Heisenberg group. Then, we turn our attention to general stratified Lie groups of step two such as H-type groups and Métivier groups.

The Heisenberg Group

Let us consider in $\mathbb{C}^d\times\mathbb{R}$ (whose points we denote by (z,t) with $t\in\mathbb{R}$ and $z=(z_1,\cdots,z_d)\in\mathbb{C}^d$) the following composition law

$$(z,t)\circ\left(z',t'\right)=\left(z+z',t+t'+2\mathrm{Im}\left(z\cdot\overline{z'}\right)\right). \tag{2.2}$$

In (2.2), we have set $\mathrm{Im}(x+iy)=y$ $(x,y\in\mathbb{R})$, whereas $z\cdot\overline{z'}$ denotes the usual Hermitian inner product in $\mathbb{C}^d$,

$$z\cdot\overline{z'}=\sum_{j=1}^{d}\left(x_j+iy_j\right)\left(x'_j-iy'_j\right).$$

Hereafter we agree to identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$ and to use the following notation to denote the points of $\mathbb{C}^d\times\mathbb{R}=\mathbb{R}^{2d+1}$:

$$(z,t)=(x,y,t)=(x_1,\cdots,x_d,y_1,\cdots,y_d,t)$$

with $z=(z_1,\cdots,z_d),z_j=x_j+iy_j$ and $x_j,y_j,t\in\mathbb{R}$. Then, the composition law $\circ$ can be explicitly written as

$$(x,y,t)\circ(x',y',t')=\big(x+x',y+y',t+t'+2\langle y,x'\rangle-2\langle x,y'\rangle\big), \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^d$. It is quite easy to verify that $\big(\mathbb{R}^{2d+1},\circ\big)$ is a Lie group whose identity is the origin and where the inverse is given by $(z,t)^{-1}=(-z,-t)$. Let us now consider the dilations

$$\delta_\lambda:\mathbb{R}^{2d+1}\to\mathbb{R}^{2d+1},\quad \delta_\lambda(z,t)=\big(\lambda z,\lambda^2 t\big). \tag{2.4}$$

A trivial computation shows that δ_λ is an automorphism of $\big(\mathbb{R}^{2d+1},\circ\big)$ for every $\lambda>0$. Then $\mathbb{H}^d=\big(\mathbb{R}^{2d+1},0,\delta_\lambda\big)$ is a homogeneous group. It is called the Heisenberg group in $\mathbb{R}^{2d+1}$.

The Jacobian basis of $\mathfrak{h}_d$, the Lie algebra of $\mathbb{H}^d$, is given by

$$X_j=\partial_{x_j}+2y_j\partial_t,\quad Y_j=\partial_{y_j}-2x_j\partial_t,\quad j=1,\cdots,d,\quad T=\partial_t.$$

Regarding the choice of suitable bases, let $(x_1,\cdots,x_d,y_1,\cdots,y_d)$ be a basis of $\mathbb{R}^{2d}$ in which the matrix of σ_c assumes the form

$$\begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}\in\mathcal{M}_{2d}(\mathbb{R}).$$

For $\lambda>0$, we choose $(x_1,\cdots,x_d,y_1,\cdots,y_d)$ as a basis of $\mathbb{R}^{2d}$, while for $\lambda<0$ this choice becomes $(y_1,\cdots,y_d,x_1,\cdots,x_d)$. Hence, for any $\lambda\in\mathbb{R}^*$, we have, as desired,

$$B^{(\lambda)}=\begin{bmatrix} 0 & 4|\lambda|I_d \\ -4|\lambda|I_d & 0 \end{bmatrix}\in\mathcal{M}_{2d}(\mathbb{R}).$$

Its radical reduces to $\{0\}$ with $\Lambda=\mathbb{R}^*$, and $\big|\eta_j(\lambda)\big|=4|\lambda|$ for all $j\in\{1,\cdots,d\}$.

H-type group

Consider the homogeneous Lie group

$$\mathbb{H}=\big(\mathbb{R}^{n+m},\circ,\delta_\lambda\big)$$

with composition law as

$$(x,t)\circ(\xi,\tau)=\left(x+\xi,t_1+\tau_1+\frac{1}{2}\langle B^{(1)}x,\xi\rangle,\cdots,t_m+\tau_m+\frac{1}{2}\langle B^{(m)}x,\xi\rangle\right)$$

where $B^{(1)},\cdots,B^{(m)}$ are fixed $n\times n$ matrices, and dilations as in (2.4). Let us also assume that the matrices $B^{(1)},\cdots,B^{(m)}$ have the following properties:

(1) $B^{(j)}$ is an $n\times n$ skew-symmetric and orthogonal matrix for every $j\leq m$;

(2) $B^{(i)}B^{(j)}=-B^{(j)}B^{(i)}$ for every $i,j\in\{1,\cdots,m\}$ with $i\neq j$.

If all these conditions are satisfied, $\mathbb{H}$ is called a group of Heisenberg-type, in short, a H-type group.

A H-type group is a stratified Lie group, since conditions (1) and (2) imply the linear independence of $B^{(1)},\cdots,B^{(m)}$. Indeed, if $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{R}^m\backslash\{0\}$, then

$$\frac{1}{|\alpha|}\sum_{s=1}^{m}\alpha_s B^{(s)}$$

is orthogonal (hence non-vanishing), as the following computation shows,

$$\begin{aligned}&\left(\frac{1}{|\alpha|}\sum_{s=1}^{m}\alpha_s B^{(s)}\right)\cdot\left(\frac{1}{|\alpha|}\sum_{s=1}^{m}\alpha_s B^{(s)}\right)^T\\&=-\frac{1}{|\alpha|^2}\sum_{r,s\leq m}\alpha_r\alpha_s B^{(r)}B^{(s)}\\&=-\frac{1}{|\alpha|^2}\sum_{r\leq m}\alpha_r^2\left(B^{(r)}\right)^2-\frac{1}{|\alpha|^2}\sum_{r,s\leq m,r\neq s}\alpha_r\alpha_s B^{(r)}B^{(s)}\\&=\mathbb{I}_n.\end{aligned}$$

Here we used the following facts: $\left(B^{(r)}\right)^2=-\mathbb{I}_n$, since $B^{(r)}$ is skew-symmetric and orthogonal; $B^{(r)}B^{(s)}=-B^{(s)}B^{(r)}$ according to condition (2).

The generators of $\mathbb{H}$ are the vector fields

$$X_i=\partial_{x_i}+\frac{1}{2}\sum_{k=1}^{m}\left(\sum_{l=1}^{n}b_{i,l}^{(k)}x_l\right)\partial_{t_k},\quad i=1,\cdots,n.$$

Moreover, if we set

$$T_k:=\partial t_k,\quad k=1,\cdots,m,$$

then we know that

$$\{X_1,\cdots,X_n;T_1,\cdots,T_m\}$$

is the Jacobian basis for $\mathbb{H}$.

Remark 2.2. The first layer of an H-type group has even dimension n. Indeed, if B is a $n\times n$ skew-symmetric orthogonal matrix, we have $\mathbb{I}_n=B\cdot B^T=-B^2$, whence $1=(-1)^n(\det B)^2$. In that case, we have $\Lambda=\mathbb{R}^m\backslash\{0\}$ with $\eta_j(\lambda)=\sqrt{\lambda_1^2+\cdots+\lambda_m^2}$ for all $j\in\{1,\cdots,l\}$.

Remark 2.3. With the previous notation, if $\mathbb{H}=(\mathbb{R}^{n+m},\circ,\delta_\lambda)$ is an H-type group, then

$$\mathfrak{z}=\{(0,t)\,|\,t\in\mathbb{R}^m\}$$

is the center of $\mathbb{H}$. Indeed, let $(y,t)\in\mathbb{H}$ be such that

$$(x,s)\circ(y,t)=(y,t)\circ(x,s) \quad \text{for every } (x,s)\in\mathbb{H}.$$

This holds iff

$$\langle B^{(k)}x,y\rangle=\langle B^{(k)}y,x\rangle$$

for any $x\in\mathbb{R}^n$ and any $k\in\{1,\cdots,m\}$. Then, since $\left(B^{(k)}\right)^T=-B^{(k)}$,

$$\langle B^{(k)}y,x\rangle=0 \quad \forall x\in\mathbb{R}^n,\ \forall k\in\{1,\cdots,m\},$$

so that $y=0$ because $B^{(k)}$ is orthogonal (hence non-singular).

Remark 2.4. The groups of Heisenberg-type were introduced by A. Kaplan in [31]. He also shows the following result. Let n,m be two positive integers. Then there exists an H-type group of dimension $n+m$ whose center has dimension m if and only if it holds $m<\rho(n)$, where ρ is the so-called Hurwitz-Radon function, i.e.

$$\rho:\mathbb{N}\to\mathbb{N}, \quad \rho(n):=8p+q, \quad \text{where } n=(\text{odd})\cdot 2^{4p+q}, \quad 0\le q\le 3.$$

We explicitly remark that if n is odd, then $\rho(n)=0$, whence the first layer of any H-type group has even dimension (as we already proved in Remark 2.2).

Métivier group

Following G. Métivier [32], we give the following definition.

Definition 2.2. *Let $\mathfrak{g}$ be a (finite-dimensional real) Lie algebra, and let us denote by $\mathfrak{z}$ its center. We say that $\mathfrak{g}$ is of Métivier Lie algebra if it admits a vector space decomposition*

$$\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2\begin{cases}[\mathfrak{g}_1,\mathfrak{g}_1]\subseteq\mathfrak{g}_2,\\ \mathfrak{g}_2\subseteq\mathfrak{z}.\end{cases}$$

with the following additional property: for every $\eta\in\mathfrak{g}_2^*$ (the dual space of $\mathfrak{g}_2$), the skew-symmetric bilinear form on $\mathfrak{g}_1$ defined by

$$B_\eta:\mathfrak{g}_1\times\mathfrak{g}_1\to\mathbb{R}, \quad B_\eta\left(X,X'\right):=\eta\left([X,X']\right)$$

is non-degenerate whenever $\eta\neq 0$.

We say that a Lie group is a Métivier group, if its Lie algebra is of Métivier Lie algebra.

Proposition 2.1. *A Métivier group is a stratified Lie group $\mathbb{G}$ of step two such that if*

$$\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2\left([\mathfrak{g}_1,\mathfrak{g}_1]=\mathfrak{g}_2,\ [\mathfrak{g}_1,\mathfrak{g}_2]=\{0\}\right)$$

is any stratification of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$, then the following property holds: for every non-vanishing linear map η from $\mathfrak{g}_2$ to $\mathbb{R}$, the (skew-symmetric) bilinear form B_η on $\mathfrak{g}_1$ defined by

$$B_\eta\left(X,X'\right):=\eta\left(\left[X,X'\right]\right),\quad X,X'\in\mathfrak{g}_1,$$

is non-degenerate.

When $\mathbb{G}$ is expressed in its logarithmic coordinates, the above definition is easily re-written as follows. We consider a homogeneous Lie group of step two $\mathbb{G}=(\mathbb{R}^{n+m},\circ,\delta_\lambda)$ with the composition law as in (2.4), i.e.

$$(x,t)\circ(\xi,\tau)=\left(x+\xi,t_1+\tau_1+\frac{1}{2}\langle B^{(1)}x,\xi\rangle,\cdots,t_m+\tau_m+\frac{1}{2}\langle B^{(m)}x,\xi\rangle\right),$$

where $B^{(1)},\cdots,B^{(m)}$ are fixed $n\times n$ matrices, and the group of dilations is $\delta_\lambda(x,t)=\left(\lambda x,\lambda^2 t\right)$. For the sake of simplicity, we may also suppose that the matrices $B^{(k)}$ are skew-symmetric.

Now, if η is a linear map from $\mathfrak{g}_2$ to $\mathbb{R}$, there exist m scalars $\eta_1,\cdots,\eta_m\in\mathbb{R}$ such that

$$\eta:\mathfrak{g}_2\to\mathbb{R},\quad \eta\left(\partial_{t_i}\right)=\eta_i\quad \text{for all } i=1,\cdots,m.$$

In particular, the map B_η can be explicitly written as follows

$$\text{if } X=\sum_{i=1}^n v_iX_i \text{ and } X'=\sum_{i=1}^n v_i'X_i,\ \text{ then } B_\eta\left(X,X'\right)=\sum_{i,j=1}^m\left(-\sum_{k=1}^m\eta_kB_{i,j}^{(k)}\right)v_iv_j'.$$

In other words, the matrix representing the (skew-symmetric) bilinear map B_η w.r.t. the basis $X_1,\cdots,X_m$ of $\mathfrak{g}_1$ is the matrix

$$\eta_1B^{(1)}+\cdots+\eta_nB^{(n)}.$$

Hence, to ask for B_η to be non-degenerate (for every $\eta\neq 0$) is equivalent to ask that any linear combination of the matrices $B^{(k)}$ is non-singular, unless it is the null matrix. We have thus obtained the following proposition.

Proposition 2.2. *Let $\mathbb{G}=(\mathbb{R}^{n+m},\circ)$ be a stratified Lie group of step two, with the composition law*

$$(x,t)\circ(\xi,\tau)=\left(x+\xi,t_1+\tau_1+\frac{1}{2}\langle B^{(1)}x,\xi\rangle,\ldots,t_m+\tau_m+\frac{1}{2}\langle B^{(m)}x,\xi\rangle\right),$$

where $B^{(1)},\cdots,B^{(m)}$ are $n\times n$ skew-symmetric linearly independent matrices. Then $\mathbb{G}$ is a Métivier group if and only if every non-vanishing linear combination of the matrices $B^{(k)}$ is non-singular.

In particular, if the above $\mathbb{G}$ is a Métivier group, then the $B^{(k)}$ are all non-singular $n\times n$ matrices, but since the $B^{(k)}$ are also skew-symmetric, this implies that n is necessarily even.

Remark 2.5 (Any H-type group is a Métivier group)**.** Indeed, as it can be seen from the definition of H-type group that, for every $\eta=(\eta_1,\cdots,\eta_n)\in\mathbb{R}^n,\eta\neq 0$, we proved that $\sum_{k=1}^m \eta_k B^{(k)}$ is $|\eta|$ times an orthogonal matrix, hence (in particular) $\sum_{k=1}^m \eta_k B^{(k)}$ is non-singular. The converse is not true. For example, consider the group on $\mathbb{R}^5$ (the points are denoted by $(x,t), x\in\mathbb{R}^4$, $t\in\mathbb{R}$) with the composition law

$$(x,t)\circ(\xi,\tau)=\left(x+\xi,t+\tau+\frac{1}{2}\langle Bx,\xi\rangle\right),$$

where

$$B=\begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2\\ 0 & 0 & -2 & 0\end{pmatrix}.$$

Then $\mathbb{G}$ is obviously a Métivier group, for B is a non-singular skew-symmetric matrix. But $\mathbb{G}$ is not a H-type group, for B is not orthogonal.

2.4 The Fourier transform

The stratified Lie groups are noncommutative, then the Fourier transform on $\mathbb{G}$ is defined using irreducible unitary representations of $\mathbb{G}$. We devote this section to the introduction of the basic concepts that will be needed in the sequel. For (λ,ν,w) in $\Lambda\times\mathbb{R}^k\times\mathbb{R}^N$ with

$$w=(x,y,r,t)\in\mathbb{R}^d\oplus\mathbb{R}^d\oplus\mathbb{R}^k\oplus\mathbb{R}^m=\mathbb{R}^N,$$

we define the irreducible unitary representations of $\mathbb{R}^N$, equipped with the group law of the nilpotent group defined above, on $L^2\left(\mathbb{R}^d\right)$

$$\left(\pi_{\lambda,\nu}\phi\right)(\xi)=e^{i\langle\nu,r\rangle}e^{i\langle\lambda,t\rangle}e^{i\left\langle\eta(\lambda)\cdot\left(\xi+\frac{1}{2}x\right),y\right\rangle}\phi(\xi+x).$$

With these notations, the Fourier transform of an integrable function of $\mathbb{G}$ is defined as follows:

Definition 2.3. *The Fourier transform of the function $f\in L^1(\mathbb{G})$ at the point*

$$(\lambda,\nu)\in\Lambda\times\mathbb{R}^k$$

is a unitary operator acting on $L^2(\mathbb{G})$ with

$$\mathcal{F}(f)(\lambda,\nu)=\hat{f}(\lambda,\nu):=\int_{\mathbb{G}} f(w)\pi_{\lambda,\nu}(w^{-1})\mathrm{d}w.$$

Proposition 2.3. *The Fourier transformation is continuous in all its variables, in the following sense.*

- *For any $\lambda\in\Lambda$ and $\nu\in\mathbb{R}^k$, the map*
$$\mathcal{F}(\cdot)(\lambda,\nu):\ L^1\left(\mathbb{R}^d\right)\longrightarrow\mathcal{L}\left(L^2\left(\mathbb{R}^d\right)\right)$$
is linear and continuous, with norm bounded by 1.
- *For any $u\in L^2\left(\mathbb{R}^d\right)$ and $f\in L^1\left(\mathbb{R}^d\right)$, the map*
$$\mathcal{F}(f)(\cdot,\cdot)(u):\ \Lambda\times\mathbb{R}^k\longrightarrow L^2\left(\mathbb{R}^d\right)$$
is continuous.

Further, the Fourier transform can be extended to an isometry from $L^2(\mathbb{G})$ onto the Hilbert space of two-parameter families $A=\{A(\lambda,v)\}_{(\lambda,v)\in\Lambda\times\mathbb{R}^k}$ of operators on $L^2\left(\mathbb{R}^d\right)$ which are Hilbert-Schmidt for almost every $(\lambda,v)\in\Lambda\times\mathbb{R}^k$, with $\|A(\lambda,v)\|_{\mathrm{HS}\left(L^2\left(\mathbb{R}^d\right)\right)}$ measurable and with norm
$$\|A\|:=\left(\iint_{\Lambda\times\mathbb{R}^k}\|A(\lambda,v)\|^2_{\mathrm{HS}\left(L^2\left(\mathbb{R}^d\right)\right)}\mathrm{Pf}(\lambda)\mathrm{d}\nu\mathrm{d}\lambda\right)^{\frac{1}{2}}<\infty,$$
where $\mathrm{Pf}(\lambda):=\prod_{j=1}^d\eta_j(\lambda)$ is the Pfaffian of $B^{(\lambda)}$. We have the following Fourier-Plancherel formula:

Proposition 2.4. *There exists some constant $\kappa>0$ depending only on the choice of the group such that, for any $f\in L^1(\mathbb{G})\cap L^2(\mathbb{G})$, there holds*
$$\int_{\mathbb{G}}|f(w)|^2\mathrm{d}w=\kappa\iint_{\Lambda\times\mathbb{R}^k}\|\mathcal{F}(f)(\lambda,\nu)\|^2_{\mathrm{HS}\left(L^2\left(\mathbb{R}^d\right)\right)}\mathrm{Pf}(\lambda)\mathrm{d}\lambda\mathrm{d}\nu.$$

Remark 2.6. On the Heisenberg group $\mathbb{H}^d$, the Pfaffian is simply $\mathrm{Pf}(\lambda)=|\lambda|^d$ and the value of κ is known, namely
$$\kappa\left(\mathbb{H}^d\right)=\frac{2^{d-1}}{\pi^{d+1}}.$$

In this context, we have an inversion formula as stated in the following proposition:

Proposition 2.5. *For $f\in L^1\left(\mathbb{R}^N\right)$ and almost every $w\in\mathbb{R}^N$, the following inversion formula holds:*
$$f(w)=\kappa\iint_{\Lambda\times\mathbb{R}^k}\mathrm{tr}\left((\pi_{\lambda,\nu}(w))^*\mathcal{F}(f)(\lambda,\nu)\right)\mathrm{Pf}(\lambda)\mathrm{d}\lambda\mathrm{d}\nu$$
with the same constant $\kappa>0$.

Finally, the Fourier transform exchanges as usual convolution and product, in the following sense.

Proposition 2.6. *For any $f_1,f_2\in L^1\left(\mathbb{R}^N\right)$ and $(\lambda,\nu)\in\Lambda\times\mathbb{R}^k$, we have, denoting by $\cdot$ the operator composition on $\mathcal{L}\left(L^2\left(\mathbb{R}^d\right)\right)$,*
$$\mathcal{F}(f_1*f_2)(\lambda,\nu)=\mathcal{F}(f_1)(\lambda,\nu)\cdot\mathcal{F}(f_2)(\lambda,\nu).$$

2.5 The sub-Laplacian operator

Let $\mathfrak{g}$ be a 2-step stratified Lie algebra with a basis $\mathcal{B}$ as before. Now we consider elements of $\mathfrak{g}$ as left invariant differential operators acting on $C^\infty(\mathbb{G})$, that is given $X \in \mathfrak{g}$ and $f \in C^\infty(\mathbb{G})$, the differential operator X acts on f by the rule

$$(Xf)(g) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} f(g \cdot \exp(sX)). \tag{2.5}$$

We define the sub-Laplacian of $\mathbb{G}$ by

$$\mathcal{L} = -\sum_{i=1}^{n} X_i^2.$$

It is a self-adjoint operator which is independent of the orthonormal basis $(X_1, \cdots, X_n)$, and homogeneous of degree 2 with respect to the dilations in the sense that

$$\delta_\lambda^{-1} \mathcal{L} \delta_\lambda = \lambda^2 \mathcal{L}.$$

To write its expression in Fourier space, we analysis the left-invariant vector fields as follows. Let $\mathfrak{g}$ be the Lie algebra of all left-invariant vector fields on $\mathbb{G}$. For $j=1,\cdots,d$, let $\gamma_{1,j}: \mathbb{R} \to \mathbb{G}$ and $\gamma_{2,j}: \mathbb{R} \to \mathbb{G}$ be curves in $\mathbb{G}$ given by

$$\gamma_{1,j}(\tau) = \big(\tau e_j, 0, 0, 0\big), \quad \gamma_{2,j}(\tau) = \big(0, \tau e_j, 0, 0\big)$$

for all $\tau \in \mathbb{R}$, where e_j is the standard unit vector in $\mathbb{R}^d$. For all $l = 1, \cdots, k$ and $s = 1, \cdots, m$, let $\gamma_{3,l}: \mathbb{R} \to \mathbb{G}$ and $\gamma_{4,s}: \mathbb{R} \to \mathbb{G}$ be curves in $\mathbb{G}$ given by

$$\gamma_{3,l}(\tau) = (0, 0, \tau e_l, 0), \quad \gamma_{4,k}(\tau) = (0, 0, 0, \tau e_s)$$

for all $\tau \in \mathbb{R}$, where e_l is the standard unit vector in $\mathbb{R}^k$ and e_s is the standard unit vector in $\mathbb{R}^m$. Then we define the left-invariant vector fields X_j, Y_j and $R_l, T_s, j=1,\cdots,d$, $l=1,\cdots,k$, $s=1,\cdots,m$, on $\mathbb{G}$ as follows. Let $f \in C^\infty(\mathbb{G})$. Then for all $j = 1, \cdots, d$, we define X_j and Y_j by

$$\begin{aligned}
\big(X_j f\big)(x,y,r,t) &= \frac{\mathrm{d}}{\mathrm{d}\tau} f\big((x,y,r,t)\cdot\gamma_{1,j}(\tau)\big)\Big|_{\tau=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau} f\left(x+\tau e_j, y, r, \left(t_s + \frac{1}{2}(B_s y, \tau e_k)\right)_{s=1}^{m}\right)\Bigg|_{\tau=0} \\
&= \frac{\partial}{\partial x_j} f(x,y,r,s) + \frac{1}{2}\sum_{s=1}^{m}\big(B_s y, e_j\big)\frac{\partial}{\partial t_s} f(x,y,r,s)
\end{aligned}$$

and

$$\begin{aligned}
\big(Y_j f\big)(x,y,r,t) &= \frac{\mathrm{d}}{\mathrm{d}\tau} f\big((x,y,r,t)\cdot\gamma_{2,j}(\tau)\big)\Big|_{\tau=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau} f\left(x, y+\tau e_j, r, \left(t_s - \frac{1}{2}(x, \tau B_k e_j)\right)_{s=1}^{m}\right)\Bigg|_{\tau=0} \\
&= \frac{\partial}{\partial y_j} f(x,y,r,s) - \frac{1}{2}\sum_{s=1}^{m}\big(x, B_k e_j\big)\frac{\partial}{\partial t_s} f(x,y,r,s)
\end{aligned}$$

for all $(x,y,r,s)\in\mathbb{G}$. Similarly, for $l=1,\cdots,k$ and $s=1,\cdots,m$, the function R_lf and T_sf are defined by

$$\begin{aligned}(R_lf)(x,y,r,t)&=\frac{\mathrm{d}}{\mathrm{d}\tau}f\left((x,y,r,t)\cdot\gamma_{3,l}(\tau)\right)\Big|_{\tau=0}\\&=\frac{\mathrm{d}}{\mathrm{d}\tau}f(x,y,r+\tau e_l,t)\Big|_{\tau=0}=\frac{\partial}{\partial r_l}(x,y,r,t)\end{aligned}$$

and

$$\begin{aligned}(T_sf)(x,y,r,t)&=\frac{\mathrm{d}}{\mathrm{d}\tau}f\left((x,y,r,t)\cdot\gamma_{4,s}(\tau)\right)\Big|_{\tau=0}\\&=\frac{\mathrm{d}}{\mathrm{d}\tau}f(x,y,r,t+\tau e_s)\Big|_{\tau=0}=\frac{\partial}{\partial t_s}f(x,y,r,t)\end{aligned}$$

for all $(x,y,r,t)\in\mathbb{G}$. We can easily check that

$$[X_i,Y_j]=-\frac{1}{4}\sum_{s=1}^{m}(B_k)_{ij}T_s,\quad i,j=1,2,\cdots,N$$

and the other commutators are zero. It follows from a theorem of Hörmander [33, Theorem 1.1] that $\mathcal{L}$ is hypoelliptic.

We can now define the sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ by

$$\mathcal{L}=-\sum_{j=1}^{d}\left(X_j^2+Y_j^2\right)-\sum_{l=1}^{k}R_l^2.$$

Explicitly,

$$\mathcal{L}=-\Delta_x-\Delta_y-\Delta_r-\frac{1}{4}\left(|x|^2+|y|^2\right)\Delta_t+\sum_{s=1}^{m}\sum_{j=1}^{d}\left\{-\left(B_sy,e_j\right)\frac{\partial}{\partial x_j}+\left(x,B_se_j\right)\frac{\partial}{\partial y_j}\right\}\frac{\partial}{\partial t_s}.$$

By taking the Fourier transform of the sub-Laplacian $\mathcal{L}$ with respect to t, we get parametrized λ-twisted sub-Laplacian $\mathcal{L}^\lambda,\lambda\in\mathbb{R}^m$, given by

$$\mathcal{L}^\lambda=-\Delta_x-\Delta_y-\Delta_r+\frac{1}{4}(|x|^2+|y|^2)|\lambda|^2-i\sum_{j=1}^{d}\left\{-\left(B^{(\lambda)}y,e_j\right)\frac{\partial}{\partial x_j}+\left(x,B^{(\lambda)}e_j\right)\frac{\partial}{\partial y_j}\right\},$$

where we use

$$B^{(\lambda)}=\sum_{s=1}^{m}\lambda_sB_s.$$

For $j=1,\cdots,d$, we define the linear partial differential operators Z_j^λ and $\bar{Z}_j^\lambda$ by

$$Z_j^\lambda=\partial_{z_j}+\frac{1}{2}i\lambda\sum_{s=1}^{m}(B_s)_j\bar{z}_j,\quad \bar{Z}_j^\lambda=\partial_{\bar{z}_j}-\frac{1}{2}i\lambda\sum_{s=1}^{m}(B_s)_jz_j.$$

Then

$$\mathcal{L}^{\lambda}=-\frac{1}{2}\sum_{j=1}^{d}\left(Z_j^{\lambda}\bar{Z}_j^{\lambda}+\bar{Z}_j^{\lambda}Z_j^{\lambda}\right)-\sum_{l=1}^{k}R_l^2=-\Delta_z-\Delta_r+\frac{1}{4}|z|^2|\lambda|^2-iN,$$

where N is the operator

$$N=i\sum_{j=1}^{d}\left\{-\left(B^{(\lambda)}y,e_j\right)\frac{\partial}{\partial x_j}+\left(x,B^{(\lambda)}e_j\right)\frac{\partial}{\partial y_j}\right\}.$$

In [30], we demonstrate the beautiful interplay between the representation theory on $\mathbb{G}$ and the classical expansions in terms of Hermite functions. If $p,q\in\mathbb{R}^d$, $\eta=(\eta_1,\cdots,\eta_d)\in(\mathbb{R}_+^*)^d$ and $\alpha\in\mathbb{N}^d$, we define the rescaled Hermite function Φ_α^λ by

$$\Phi_\alpha^\lambda:=|\eta|^{\frac{d}{4}}\Phi_\alpha\left(|\eta|^{\frac{1}{2}}\cdot\right),$$

and the special Hermite functions

$$\Phi_{\alpha,\beta}^{\lambda}(x)=\mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{i\eta(\lambda)\cdot px}\Phi_\alpha^\lambda\left(x+\frac{q}{2}\right)\overline{\Phi_\beta^\lambda\left(x-\frac{q}{2}\right)}\mathrm{d}x.$$

In particular, they form an orthonormal basis of $L^2\left(\mathbb{R}^d\right)$ and we have the rescaled harmonic oscillator

$$\mathcal{H}_\lambda\Phi_\alpha^\lambda:=(-\Delta+|\eta\cdot x|^2)\Phi_\alpha^\lambda=\sum_{j=1}^{d}\eta_j(\lambda)(2\alpha_j+1)\Phi_\alpha^\lambda,$$

which can help us prove that $\Phi_{\alpha,\beta}^\lambda$ are eigenfunctions of the λ-twisted sub-Laplacian $\mathcal{L}^\lambda$.

Theorem 2.1. ([30]) *For $\lambda\in\Lambda,\nu\in\mathbb{R}^k$, one has the formula*

$$\mathcal{L}^{\lambda}(\Phi_{\alpha,\beta}^{\lambda})=\left(\sum_{j=1}^{d}\eta_j(\lambda)(2\alpha_j+1)+\sum_{j=1}^{k}\nu_j^2\right)\Phi_{\alpha,\beta}^{\lambda}.$$

3 Essential self-adjointness and spectrum

In this section we look at the sub-Laplacian $\mathcal{L}$ as an unbounded linear operator from $L^2(\mathbb{G})$ into $L^2(\mathbb{G})$ with dense domain given by the Schwartz space $\mathcal{S}(\mathbb{G})$. Since $-\bar{Z}_j^\lambda$ is the formal adjoint of Z_j^λ, it follows that the λ-twisted sub-Laplacian $\mathcal{L}^\lambda$ is a symmetric operator from $L^2(\mathbb{G})$ into $L^2(\mathbb{G})$ with dense domain $\mathcal{S}(\mathbb{G})$. As such, it is closable and we denote the closure by $\mathcal{L}_0^\lambda$.

Lemma 3.1. *$\mathcal{L}_0^\lambda$ is closed and symmetric.*

Proof. It's easy to see that $\mathcal{L}_0^\lambda$ is closed. We only need to prove that $\mathcal{L}_0^\lambda$ is symmetric. Let u and v be functions in the domain $\mathcal{D}\left(\mathcal{L}_0^\lambda\right)$ of $\mathcal{L}_0^\lambda$. Then we can find sequences $\{\varphi_l\}_{l=1}^\infty$ and $\{\psi_l\}_{l=1}^\infty$ in $\mathcal{S}$ such that

$$\varphi_l \to u, \quad \mathcal{L}^\lambda \varphi_l \to \mathcal{L}_0^\lambda u,$$

and

$$\psi_l \to v, \quad \mathcal{L}^\lambda \psi_l \to \mathcal{L}_0^\lambda v,$$

in $L^2(\mathbb{G})$ as $l \to \infty$. So, using the symmetry of $\mathcal{L}^\lambda$ as a linear operator from $L^2(\mathbb{G})$ into $L^2(\mathbb{G})$ with domain $\mathcal{S}$,

$$\left(\mathcal{L}_0^\lambda u, v\right) = \lim_{l\to\infty} \left(\mathcal{L}^\lambda \varphi_l, \psi_l\right) = \lim_{l\to\infty} \left(\varphi_l, \mathcal{L}^\lambda \psi_l\right) = \left(u, \mathcal{L}_0^\lambda v\right).$$

Therefore $\mathcal{L}_0^\lambda$ is symmetric. □

Theorem 3.1. *$\mathcal{L}_0^\lambda$ is self-adjoint.*

Proof. Since $\mathcal{L}_0^\lambda$ is closed and symmetric, it follows that $\mathcal{L}_0^\lambda$ is self-adjoint if we can prove that the resolvent set $\rho\left(\mathcal{L}_0^\lambda\right)$ of $\mathcal{L}_0^\lambda$ contains a real number. (See, for instance, [34].) It is sufficient to prove that the range $R\left(\mathcal{L}_0^\lambda\right)$ of $\mathcal{L}_0^\lambda$ is dense in $L^2(\mathbb{G})$ and there exists a positive constant C such that

$$\left\|\mathcal{L}_0^\lambda u\right\|_2 \geq C\|u\|_2, \quad u \in \mathcal{D}(\mathcal{L}_0^\lambda). \tag{3.1}$$

The density follows from the global hypoellipticity of $\mathcal{L}^\lambda$ in the sense of Schwartz functions and distributions given in [35]. To prove the inequality (3.1), let $\varphi \in \mathcal{S}(\mathbb{G})$. Then, by Parseval's identity,

$$\begin{aligned}
\|\mathcal{L}^\lambda \varphi\|_2^2 &= \left\| \left(\sum_{j=1}^d \eta_j(\lambda)(2\alpha_j+1) + \sum_{j=1}^k \nu_j^2 \right) (\varphi, \Phi_{\alpha,\beta}^\lambda) \Phi_{\alpha,\beta}^\lambda \right\|^2 \\
&= \left(\sum_{j=1}^d \eta_j(\lambda)(2\alpha_j+1) + \sum_{j=1}^k \nu_j^2 \right)^2 \left| (\varphi, \Phi_{\alpha,\beta}^\lambda) \right|^2 \\
&\geq \|\varphi\|_2^2.
\end{aligned}$$

□

Corollary 3.1. *The sub-Laplacian $\mathcal{L}$ from $L^2(\mathbb{G})$ into $L^2(\mathbb{G})$ with dense domain $\mathcal{S}(\mathbb{G})$ is essentially self-adjoint.*

Let A be a closed linear operator from a complex Banach space X into X with dense domain $\mathcal{D}(A)$. Then the resolvent set $\rho(A)$ of A is defined to be the set of all complex numbers λ for which $A - \lambda I : \mathcal{D}(A) \to X$ is bijective, where I is the identity operator on X. The spectrum $\Sigma(A)$ is simply the complement of $\rho(A)$ in $\mathbb{C}$.

Following [36], the point spectrum $\Sigma_p(A)$ of A is the set of all complex numbers λ such that $A - \lambda I$ is not injective. The continuous spectrum $\Sigma_c(A)$ of A is the set of all complex numbers λ such that the range $R(A-\lambda I)$ of $A-\lambda I$ is dense in $X,(A-\lambda I)^{-1}$ exists, but is unbounded. The residual spectrum $\Sigma_r(A)$ of A is the set of all complex numbers λ such that $(A-\lambda I)^{-1}$ is bounded, but the range $R(A-\lambda I)$ is not dense in X. It is easy to see that $\Sigma_p(A)$,$\Sigma_c(A)$ and $\Sigma_r(A)$ are mutually disjoint and

$$\Sigma(A)=\Sigma_p(A)\cup\Sigma_c(A)\cup\Sigma_r(A).$$

Moreover, it is well-known that if A is a self-adjoint operator on a complex and separable Hilbert space X, then

$$\Sigma_r(A)=\emptyset.$$

Then we have the precise description of the spectrum of the sub-Laplacian on $\mathbb{G}$.

Theorem 3.2. $\Sigma(\mathcal{L}_0)=\Sigma_c(\mathcal{L}_0)=[0,\infty)$.

Proof. Since $\mathcal{L}_0$ is self-adjoint, it follows that

$$\Sigma(\mathcal{L}_0)=\Sigma_c(\mathcal{L}_0).$$

Next we prove that $\mathcal{L}_0$ has no eigenvalues in $[0,\infty)$. It follows from Liouville's Theorem ([35]) that 0 is not an eigenvalue of $\mathcal{L}_0$. Then, let τ be a positive number such that there exists a function u in $L^2(\mathbb{G})$ for which

$$\mathcal{L}_0 u=\tau u.$$

By taking the Fourier transform with respect to t, we get

$$L^\lambda u^\lambda=\tau u^\lambda,$$

where u^λ is the Fourier transform of u in t direction. But it follows from Theorem 2.1 that $u^\lambda=0$ for all $\lambda\in\mathbb{R}^m\setminus\{0\},\nu\in\mathbb{R}^k\setminus\{0\}$ with

$$\tau\neq\sum_{j=1}^{d}\eta_j(\lambda)(2\alpha_j+1)+\sum_{j=1}^{k}\nu_j^2,\quad \alpha_j\in\mathbb{N}.$$

This proves that $u=0$ and hence we get a contradiction.

So, it remains to prove that $\mathcal{L}_0-\tau I$ is not surjective for all τ in $[0,\infty)$. Suppose that $\mathcal{L}_0-\tau_0 I$ is surjective for some τ_0 in $[0,\infty)$. Then λ_0 is in the resolvent set $\rho(\mathcal{L}_0)$ of $\mathcal{L}_0$. Hence there exists an open interval I_{τ_0} such that $\tau_0\in I_{\tau_0}$ and $I_{\tau_0}\subset\rho(\mathcal{L}_0)$. Let f be the function on $\mathbb{G}$ defined by

$$f(w)=h(x,y,r)e^{-\frac{t^2}{2}},\quad w\in\mathbb{G}$$

where h is an arbitrary function in $L^2\left(\mathbb{R}^{2d+k}\right)$. Then for all τ in I_{τ_0}, we can find a function u_τ in $L^2(\mathbb{G})$ such that

$$(\mathcal{L}_0-\tau I)u_\tau=f.$$

Taking the inverse Fourier transform with respect to t, we get

$$\left(L^\lambda-\tau I\right)u_\tau^\lambda=he^{-\frac{\lambda^2}{2}}$$

for almost all λ in $\mathbb{R}^m\backslash\{0\}$. So, $L^\lambda-\tau I$ is surjective for all λ in a set S_τ for which the Lebesgue measure of $\mathbb{R}^m\backslash S_\lambda$ is zero. Now, let $\lambda\in\bigcap_{r\in I_\tau}S_r$, where $\mathbb{Q}$ is the set of all rational numbers. Then $L^\lambda-\tau I$ is surjective and hence injective for all τ in $I_{\tau_0}\cap\mathbb{Q}$. Hence $L^\lambda-\tau I$ is bijective for all τ in I_{τ_0} by the fact that the resolvent set of L^λ is an open set. On the other hand, $L^\lambda-\tau I$ is one to one if and only if

$$\tau\neq\sum_{j=1}^{d}\eta_j(\lambda)(2\alpha_j+1)+\sum_{j=1}^{k}\nu_j^2,\quad \alpha_j\in\mathbb{N}.$$

This is a contradiction if we choose λ in $\bigcap_{r\in I_{\tau_0}}S_r$ to be a sufficiently small number such that $\sum_{j=1}^d\eta_j(\lambda)(2\alpha_j+1)+\sum_{j=1}^k\nu_j^2\in I_{\tau_0}$ for some nonnegative integer α_j. $\square$

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References

[1] Rudin W., Fourier Analysis on Groups. Wiley Classics Library. John Wiley & Sons, Inc., Ne -w York, 1990.
[2] Folland G. B., Stein E. M., Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg gro -up. *Comm. Pure Appl. Math.*, **27** (1974), 429-522.
[3] Fischer V., Ruzhansky M., Quantization on Nilpotent Lie Groups, volume 314 of *Progress in Mathematics.* Birkhäuser/Springer, [Cham], 2016.
[4] Folland G. B., Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat.*, **13**(2) (1975), 161-207.
[5] Goodman R. W., Nilpotent Lie Groups: Structure and Applications to Analysis. Lecture Notes in Mathematics, Vol. 562. Springer-Verlag, Berlin-New York, 1976.
[6] Rothschild L. P., Stein E. M., Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, **137** (3-4) (1976), 247-320.
[7] Taylor M. E., Noncommutative Harmonic Analysis, volume 22 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1986.
[8] Stein E. M., Harmonic Analysis: Real-variable Methods, Orthogonality, And Oscillatory Integrals, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[9] Thangavelu S., Harmonic Analysis On The Heisenberg Group, volume 159 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1998.
[10] Chang D., Kang Q. and Wang W., The heat kernel of sub-Laplace operator on nilpotent Lie groups of step two. *Appl. Anal.*, **100** (1) (2021), 17-36.
[11] Duan X., The heat kernel and Green function of the sub-Laplacian on the Heisenberg group. In *Pseudo-differential operators, generalized functions and asymptotics*, volume 231 of *Oper. Theory Adv. Appl.*, pages 55-75. Birkhäuser/Springer Basel AG, Basel, 2013.
[12] Li H., Zhang Y., Revisiting the heat kernel on isotropic and nonisotropic Heisenberg groups. *Comm. Partial Differential Equations*, **44** (6) (2019), 467-503.
[13] Molahajloo S., Wong M. W., Heat kernels and Green functions of sub-Laplacians on Heisenberg groups with multi-dimensional center. *Math. Model. Nat. Phenom.*, **13** (4) (2018), 1-14.
[14] Yang Q., Zhu F., The heat kernel on H-type groups. *Proc. Amer. Math. Soc.*, **136** (4) (2008), 1457-1464.
[15] Yang Z., Weyl transform and Heat kernel of sub-Laplacian on 2-step stratified Lie groups wi -thout the Moore-Wolf condition. *prepared*, 2022.
[16] Escobar J., On the spectrum of the Laplacian on complete Riemannian manifolds. *Comm. Par -tial Differential Equations*, **11** (1) (1986), 63-85.
[17] Escobar J., Freire A., The spectrum of the Laplacian of manifolds of positive curvature. *Duke Math. J.*, **65** (1) (1992), 1-21.
[18] Gordon C. S., Wilson E. N., The spectrum of the Laplacian on Riemannian Heisenberg manifolds. *Michigan Math. J.*, **33**(2) (1986), 253-271.
[19] Gornet R., The marked length spectrum vs. the Laplace spectrum on forms on Riemannian nilmanifolds. *Comment. Math. Helv.*, **71**(2) (1996), 297-329.
[20] Ikeda A., On the spectrum of a Riemannian manifold of positive constant curvature. *Osaka Math. J.*, **17** (1) (1980), 75-93.
[21] Ikeda A., On the spectrum of a Riemannian manifold of positive constant curvature. II. *Osaka Math. J.*, **17**(3) (1980), 691-702.
[22] Furutani K., Sagami K. and Ôtsuki N., The spectrum of the Laplacian on a certain nilpotent Lie group. *Comm. Partial Differential Equations*, **18**(3-4) (1993), 533-555.
[23] Dasgupta A., Molahajloo S. and Wong M., The spectrum of the sub-Laplacian on the Heisenberg group. *Tohoku Math. J. (2)*, **63**(2) (2011), 269-276.
[24] Corwin L. J., Greenleaf F. P., Representations of Nilpotent Lie Groups and Their Applications. Part I, volume 18 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990. Basic theory and examples.
[25] Ray S. K., Uncertainty principles on two step nilpotent Lie groups. *Proc. Indian Acad. Sci. Ma -th. Sci.*, **111**(3) (2001), 293-318.
[26] Yang Z., Weyl-Hörmander symbolic calculus on step two nilpotent Lie groups I: Basic properties. *prepared*, 2022.
[27] Varadarajan V. S., An Introduction to Harmonic Analysis on Semisimple Lie Groups, volume 16 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. Corrected reprint of the 1989 original.
[28] Moore C. C., Wolf J. A., Square integrable representations of nilpotent groups. *Trans. Amer. Math. Soc.*, **185** (1973), 445-462.
[29] Müller D., Ricci F., Solvability for a class of doubly characteristic differential operators on 2-step nilpotent groups. *Ann. of Math. (2)*, **143**(1) (1996), 1-49.
[30] Yang Z., Harmonic Analysis on 2-step Stratified Lie Groups. University of Science and Tech-

nology of China Press, Hefei, 2023.

[31] Kaplan A., Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Trans. Amer. Math. Soc.*, **258** (1) (1980), 147-153.

[32] Métivier G., Hypoellipticité analytique sur des groupes nilpotents de rang 2. *Duke Math. J.*, **47**(1) (1980), 195-221.

[33] Hörmander L., Hypoelliptic second order differential equations. *Acta Math.*, **119** (1967), 147-171.

[34] Reed M., Simon B., Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-adjointness. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[35] Yang Z., Green function and Liouville's theorem on 2-step stratified Lie groups without the Moore-Wolf condition. *prepared*, 2024.

[36] Yosida K., Functional Analysis. Classics in Mathematics. Springer-Verlag, Berlin, 1995.