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The Method of Moving Planes for Integral Equation in an Extremal Case

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Abstract. In this paper, we study the symmetry results and monotonicity of solutions for an integral equation

$$u(x)=-c_N\int_{\mathbb{R}^N}e^{u(y)}\log|x-y|\mathrm{d}y$$

in an external case.

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Key Words: Integral equation; radial symmetry; the method of moving planes.

1 Introduction

In order to study the radial symmetry and monotonicity of solutions for non-linear elliptic equations in $\mathbb{R}^N$, Gidas et al. [1] settled these two properties for Schrödinger equations by the method of moving planes, which has made great advances by [2–5] in the last decades. Recently, the authors in [6,7] obtained the symmetry result of solutions for fractional elliptic equations by adding an appropriate truncation argument in the method of moving planes to overcome the difficulty which comes from the nonlocal characteristic of the fractional Laplacian.

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Chen et al. in [8] extended the method of moving planes to solve the symmetry property of the integral equation

$$u(x)=c_{N,\alpha}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-\alpha}}u^{\frac{N+\alpha}{N-\alpha}}(y)\mathrm{d}y, \qquad x\in\mathbb{R}^N, \tag{1.1}$$

where $\alpha\in(0,N)$ and $c_{N,\alpha}$ is the normalized constant of corresponding fundamental solution. More discussion see references [9–11]. In fact, the integral equation (1.1) is equivalent to the partial differential equation

$$(-\Delta)^{\frac{\alpha}{2}}u=u^{\frac{N+\alpha}{N-\alpha}}, \qquad \text{in } \mathbb{R}^N. \tag{1.2}$$

We observe that the fundamental solution plays an important role in the equivalent between (1.1) and (1.2). However, in the extremal case that $\alpha=N$, the fundamental solution of

$$(-\Delta)^{\frac{\alpha}{2}}u=\delta_0, \qquad \text{in } \mathbb{R}^N, \tag{1.3}$$

is the logicism function

$$\Gamma_\alpha(x)=-c_N\log|x|, \qquad x\in\mathbb{R}^N, \tag{1.4}$$

which changes signs, where δ_0 is the Dirac mass concentrated at the origin and $c_N>0$ is the normalized constant.

Our interest in the present paper is to develop the method of moving planes to study radial symmetry and monotonicity of solutions for the following integral equation

$$\begin{cases} u(x)=-c_N\displaystyle\int_{\mathbb{R}^N}e^{u(y)}\log|x-y|\mathrm{d}y, \quad x\in\mathbb{R}^N,\\ \displaystyle\int_{|x|>1}e^{u(x)}\log|x|\mathrm{d}x<+\infty,\\ \displaystyle\lim_{|x|\to+\infty}e^{u(x)}=0, \end{cases} \tag{1.5}$$

where $N\geq 1$. Our main result states as following.

Theorem 1.1. *Let $u\in C(\mathbb{R}^N)$ be a solution of equation* (1.5)*. Then u is radially symmetric and strictly decreasing about some point in $\mathbb{R}^N$.*

Using the Fourier transform, we know that the problem (1.5) is equivalent to

$$\begin{cases} (-\Delta)^{\frac{N}{2}}u=e^u, \qquad \text{in } \mathbb{R}^N,\\ \displaystyle\int_{|x|>1}e^{u(x)}\log|x|\mathrm{d}x<+\infty,\\ \displaystyle\lim_{|x|\to+\infty}e^{u(x)}=0, \end{cases} \tag{1.6}$$

where $(-\Delta)^{\frac{N}{2}}$ is defined as

$$\widehat{(-\Delta)^{\frac{N}{2}}u}(\xi)=|\xi|^N\widehat{u}, \qquad u\in C_c^\infty(\mathbb{R}^N),$$