Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2015.m1324

Vol. **8**, No. 4, pp. 582-604
November 2015

Semi-Discrete and Fully Discrete Hybrid Stress Finite Element Methods for Elastodynamic Problems

Zhengqin Yu and Xiaoping Xie*

School of Mathematics, Sichuan University, Chengdu 610064, China.

Received 10 August 2013; Accepted (in revised version) 05 September 2014

Abstract. This paper proposes and analyzes semi-discrete and fully discrete hybrid stress finite element methods for elastodynamic problems. A hybrid stress quadrilateral finite element approximation is used in the space directions. A second-order center difference is adopted in the time direction for the fully discrete scheme. Error estimates of the two schemes, as well as a stability result for the fully discrete scheme, are derived. Numerical experiments are done to verify the theoretical results.

AMS subject classifications: 65N12; 65N15;65N30

Key words: elastodynamic problem, hybrid stress finite element, semi-discrete, fully discrete.

1. Introduction

Consider the following vibration model of plane elasticity:

$$\begin{cases} \mathbf{u}_{tt} - \mathbf{div}\sigma = \mathbf{f}(\mathbf{x}, t), & (\mathbf{x}, t) \in \Omega \times [0, T], \\ \sigma = \mathbb{C}\epsilon(\mathbf{u}) = 2\mu\epsilon(\mathbf{u}) + \lambda \mathrm{div}\mathbf{u}\, \mathbf{I}, & (\mathbf{x}, t) \in \Omega \times [0, T], \\ \mathbf{u} = 0, & (\mathbf{x}, t) \in \Gamma \times [0, T], \\ \mathbf{u}(\mathbf{x}, 0) = \varphi_0(\mathbf{x}), \mathbf{u}_t(\mathbf{x}, 0) = \varphi_1(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \Re^2$ is a bounded open set with boundary Γ, T a positive constant, $\mathbf{u} = (u_1, u_2)^T$ the displacement field, $\sigma = (\sigma_{ij})_{2\times 2}$ the symmetric stress tensor. $\epsilon(\mathbf{u}) = \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)/2$ the strain tensor. λ, μ are the Lamé parameters given by

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

for plane strain problems and $\lambda = \frac{E\nu}{(1+\nu)(1-\nu)}$ for plane stress problems,with $0 < \nu < 0.5$ the Poisson ratio and E the Young's modulus. $\mathbf{f} \in (L^2(\Omega \times [0, T]))^2$ denotes the body force, initial data $\varphi_0(\mathbf{x}), \varphi_1(\mathbf{x}) \in (H_0^1(\Omega))^2$.

*Corresponding author. *Email address:* zhengqinyulm@sina.com (Z. Yu), xpxie@scu.edu.cn (X. Xie)

Numerical simulation of time-dependent problems plays an important role in vibration analysis of elastic structures ([7], [21]). Basically, the corresponding discretization approach consists of getting a semi-discrete scheme by means of the finite element method in the space direction and then discretizing time derivatives to obtain a fully discrete scheme [15]. Douglas and Gupta [8] discussed the superconvergence for a mixed finite element method of wave propagation in a plane domain, where a quasi-projection analysis was given to obtain error estimates in Sobolev spaces of nonpositive index. In [1, 2], B$\acute{e}$cache, Joly and Tsogka changed the second-order system (1.1) to a first order system, and constructed a new family of quadrangular or cubic mixed finite elements for the new system. In [9], Lai, Huang and Chen established a new C^0-continuous time stepping displacement-type finite element method to solve the vibration problem of plane elasticity. Boulaajine, Farhloul and Paquet [3, 4] used a new dual mixed finite element method to solve the linear elastodynamic system with explicit/implicit Newmark schemes for the time discretizaton. The explicit Newmark scheme was shown to be stable under an appropriate CFL condition and the implicit Newmark scheme was unconditionally stable. In [6], Cheng and Xie considered a space-time nonconforming finite element method for the vibration problem.

The assumed-stress hybrid approach, pioneered by Pian [11], is a kind of mixed finite element method based on the Hellinger-Reissner variational principle which includes unknowns of displacements and stresses. In [12] Pian and Sumihara derived the famous hybrid stress element (abbr. PS) through a rational choice of stress terms. Pian and Tong [13] discussed the similarity and basic difference between the incompatible displacement model and the hybrid stress model. In the direction of determining the optimal stress parameters, there have been many other research efforts, cf. [14, 16–18, 22]. Xie and Zhou [17, 18] derived robust 4-node hybrid stress quadrilateral elements by optimizing stress modes with a so-called energy-compatibility condition, i.e. the assumed stress terms are L^2-orthogonal to the enhanced strains caused by Wilson bubble displacements. Yu, Xie and Carstensen [19] derived uniform convergence and a posteriori error estimation for the hybrid stress methods.

In this paper, we shall apply the hybrid stress finite element method to the elastodynamic system (1.1) to obtain a semi-discrete scheme, then use a second-order center difference scheme in the time direction to get a fully discrete scheme. We will derive error estimates for the two schemes and a stability result for the fully discrete explicit scheme.

The paper is organized as follows. Section 2 gives notations and weak formulations. Section 3 is devoted to the error analysis of the semi-discrete hybrid stress method. In Section 4, we show a stability result and the error estimation of the fully discrete hybrid stress scheme. Some numerical results are provided in the last section.

2. Notations and weak formulations

Throughout this paper, we denote by $H^r(\Omega)$ the standard Sobolev spaces with norm $\|\cdot\|_r$ and semi-norm $|\cdot|_r$, and $H^0 = L^2$ is the space of square integrable functions. For