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Finite Neuron Method and Convergence Analysis

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Abstract. We study a family of H^m-conforming piecewise polynomials based on the artificial neural network, referred to as the *finite neuron method* (FNM), for numerical solution of $2m$-th-order partial differential equations in $\mathbb{R}^d$ for any $m,d\geq 1$ and then provide convergence analysis for this method. Given a general domain $\Omega\subset\mathbb{R}^d$ and a partition $\mathcal{T}_h$ of Ω, it is still an open problem in general how to construct a conforming finite element subspace of $H^m(\Omega)$ that has adequate approximation properties. By using techniques from artificial neural networks, we construct a family of H^m-conforming functions consisting of piecewise polynomials of degree k for any $k\geq m$ and we further obtain the error estimate when they are applied to solve the elliptic boundary value problem of any order in any dimension. For example, the error estimates that $\|u-u_N\|_{H^m(\Omega)}=\mathcal{O}(N^{-\frac{1}{2}-\frac{1}{d}})$ is obtained for the error between the exact solution u and the finite neuron approximation u_N. A discussion is also provided on the difference and relationship between the finite neuron method and finite element methods (FEM). For example, for the finite neuron method, the underlying finite element grids are not given a priori and the discrete solution can be obtained by only solving a non-linear and non-convex optimization problem. Despite the many desirable theoretical properties of the finite neuron method analyzed in the paper, its practical value requires further investigation as the aforementioned underlying non-linear and non-convex optimization problem can be expensive and challenging to solve. For completeness and the convenience of the reader, some basic known results and their proofs.

AMS subject classifications: 65D07, 65D15, 65N22, 65N30

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1 Introduction

This paper is devoted to the study of numerical methods for high-order partial differential equations in any dimension using appropriate piecewise polynomial function classes.

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In this introduction, we will briefly describe a class of elliptic boundary value problems of any order in any dimension. We will then give an overview of some existing numerical methods for this model and other related problems. We will then explain the motivation and objective of this paper.

1.1 Model problem

Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. For any integer $m\geq 1$, we consider the following model $2m$-th-order partial differential equation with certain boundary conditions:

$$\begin{cases} Lu=f & \text{in } \Omega, \\ B^k(u)=0 & \text{on } \partial\Omega \ \ (0\leq k\leq m-1), \end{cases} \tag{1.1}$$

where L is the partial differential operator

$$Lu=\sum_{|\alpha|=m}(-1)^m\partial^\alpha(a_{\boldsymbol{\alpha}}(x)\partial^\alpha u)+a_0(x)u, \tag{1.2}$$

and $\boldsymbol{\alpha}$ denotes the n-dimensional multi-index $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_n)$ with

$$|\boldsymbol{\alpha}|=\sum_{i=1}^n\alpha_i,\quad \partial^{\boldsymbol{\alpha}}=\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}.$$

For simplicity, we assume that a_α are strictly positive and smooth functions on Ω for $|\alpha|=m$ and $\alpha=0$, namely, $\exists\alpha_0>0$, such that

$$a_\alpha(x),a_0(x)\geq\alpha_0,\quad \forall x\in\Omega,\quad |\alpha|=m. \tag{1.3}$$

Given a nonnegative integer k and a bounded domain $\Omega\subset\mathbb{R}^d$, let

$$H^k(\Omega):=\left\{v\in L^2(\Omega),\,\partial^\alpha v\in L^2(\Omega),\,|\alpha|\leq k\right\}$$

be standard Sobolev spaces with the norm and seminorm given respectively by

$$\|v\|_k:=\left(\sum_{|\alpha|\leq k}\|\partial^\alpha v\|_0^2\right)^{1/2},\quad |v|_k:=\left(\sum_{|\alpha|=k}\|\partial^\alpha v\|_0^2\right)^{1/2}.$$

For $k=0$, $H^0(\Omega)$ is the standard $L^2(\Omega)$ space with the inner product denoted by $(\cdot,\cdot)$. Similarly, for any subset $K\subset\Omega$, the $L^2(K)$ inner product is denoted by $(\cdot,\cdot)_{0,K}$. We note that, by the well-known property of the Sobolev space, the assumption (1.3) implies that

$$a(v,v)\gtrsim\|v\|_{m,\Omega}^2,\quad \forall v\in H^m(\Omega). \tag{1.4}$$

The boundary value problem (1.1) can be cast into an equivalent optimization or a variational problem as described below for some approximate subspace $V\subset H^m(\Omega)$.